

THE CLASSIFICATION OF LINKED 3-MANIFOLDS IN 6-SPACE.

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ABSTRACT. Let M_1 and M_2 be closed connected orientable 3-manifolds. We classify the sets of smooth and piecewise linear isotopy classes of embeddings $M_1 \sqcup M_2 \rightarrow S^6$.

CONTENTS

1. Introduction.	1
2. Proof of the main theorem modulo lemmas.	12
3. Proof of Surjectivity, Bijectivity, and Preimage Lemmas 2.3, 2.7, 2.9.	23
4. Proof of Calculation Lemma 2.11.	26
5. Proof of Claim 2.8 and Linking Lemma 2.12.	30
References	33

1. INTRODUCTION.

1.1. Statement of the result. All maps and manifolds in the text are smooth¹ unless specifically stated otherwise.

For a manifold N denote by $E^m(N)$ the set of isotopy classes of embeddings $N \rightarrow S^m$. The main result of the paper is Theorem 1.11 giving a classification of $E^6(M_1 \sqcup M_2)$ for arbitrary closed connected orientable 3-manifolds M_1 and M_2 . As a corollary we also get a piecewise linear (PL) classification, see Theorem 1.18 in §1.2.

We start with the previously known classifications of $E^6(S^3 \sqcup S^3)$ and $E^6(N)$, where N is a closed connected orientable 3-manifold. These results are later used in our proofs. In §1.3 we also give a brief general survey on embeddings classification.

An embedding $g : S^3 \rightarrow S^6$ is called *trivial* if it is isotopic to the standard embedding. The isotopy class of a trivial embedding is also called *trivial*. The embedded connected sum operation $\#$ (see §1.4) defines a group structure on $E^6(S^3)$. Operation $\#$ also defines an action of $E^6(S^3)$ on $E^6(N)$ for any closed connected orientable 3-manifold N .

Theorem 1.1 (A. Haefliger). $E^6(S^3) \cong \mathbb{Z}$.

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¹In this paper “smooth” means C^1 -smooth. For each C^∞ -manifold N the forgetful map from the set of C^∞ -isotopy classes of C^∞ -embeddings $N \rightarrow \mathbb{R}^m$ to the set of C^1 -isotopy classes of C^1 -embeddings $N \rightarrow \mathbb{R}^m$ is a 1-1 correspondence, see [Zh16], c.f. [Sk15, footnote 2].

Let

$$r : E^6(S^3) \rightarrow \mathbb{Z}$$

be one (of the two) isomorphisms $E^6(S^3) \rightarrow \mathbb{Z}$. We call the chosen isomorphism r the *Haefliger invariant*².

Remark 1.2. The zero of the group $E^6(S^3)$ is the trivial class. I.e., Theorem 1.1 implies that $r(g) = 0$ if and only if $g : S^3 \rightarrow S^6$ is trivial.

All the homology groups in the text are with coefficients in \mathbb{Z} unless another group is explicitly specified. For any closed connected orientable 3-manifold N the *Whitney invariant*

$$W : E^6(N) \rightarrow H_1(N)$$

is defined in [Sk08a]. We give an equivalent definition in §1.6.

For an element $a \neq 0$ of a free abelian group G denote by $\text{div}(a)$ the *divisibility* of a . I.e., $\text{div}(a)$ is the maximal positive integer such that $a = \text{div}(a)b$ for some $b \in G$. Put $\text{div}(0) = 0$. For an element a of an abelian group G denote by $\text{div}(a)$ the divisibility of the projection of a to the free part of G .

Theorem 1.3 (A. Skopenkov, others).³ *For any closed connected orientable 3-manifold N*

(I) *the Whitney invariant*

$$W : E^6(N) \rightarrow H_1(N)$$

is surjective.

(II) *The embedded connected sum action of $E^6(S^3)$ is transitive on each of the preimages of W .*

(III) *For any $[f] \in E^6(N)$ and $[g] \in E^6(S^3)$ we have that $[f] \# [g] = [f]$ if and only if the Haefliger invariant $r(g)$ is a multiple of the divisibility of the Whitney invariant $W(f)$, i.e., $r(g) = k \text{div}(W(f))$ for some integer k .*

Corollary 1.4. *Suppose that $H_1(N)$ is infinite. Then there is an element $[f] \in E^6(N)$ and a non-trivial element $[g] \in E^6(S^3)$ such that $[f] \# [g] = [f]$.*

An embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ is called *unlinked* if its components lie in pairwise disjoint balls. An unlinked embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ is called *trivial* if its restriction to each component is trivial. The isotopy class of a trivial (resp. unlinked) embedding is also called *trivial* (resp. *unlinked*). An unlinked embedding differs from a trivial embedding only by the “knotting” of the components. The component-wise embedded connected sum operation $\#$ (see §1.4) defines a group structure on $E^6(S^3 \sqcup S^3)$ and an action of $E^6(S^3 \sqcup S^3)$ on $E^6(M_1 \sqcup M_2)$ for arbitrary closed connected orientable 3-manifolds M_1 and M_2 .

For $k \in \{1, 2\}$ let

$$r_k : E^6(S^3 \sqcup S^3) \rightarrow \mathbb{Z}$$

be the Haefliger invariant of the restriction to the k -th connected component. The (defined later in §1.8) isotopy invariants

$$\lambda_1, \lambda_2 : E^6(S^3 \sqcup S^3) \rightarrow \mathbb{Z}$$

²For arbitrary closed connected orientable 3-manifold N there is a generalized version $E^6(N) \rightarrow \mathbb{Z}$ of this invariant due to M. Kreck.

³Part (III) of the Theorem is due to A. Skopenkov, see [Sk08a]. Parts (I) and (II) were known earlier, see [Sk08a, Footnote 3].

are called the (generalized) *linking coefficients*.

Denote

$$\widetilde{\mathbb{Z}}^4 := \{(a, b) \in \mathbb{Z}^2 \mid a \equiv b \pmod{2}\} \times \mathbb{Z}^2 \subset \mathbb{Z}^4.$$

Theorem 1.5 (A. Haefliger, [Ha62a]). *The map $\lambda_1 \times \lambda_2 \times r_1 \times r_2 : E^6(S^3 \sqcup S^3) \rightarrow \mathbb{Z}^4$ is a monomorphism and its image is $\widetilde{\mathbb{Z}}^4$.*

Remark 1.6. The zero of the group $E^6(S^3 \sqcup S^3)$ is the trivial class. I.e., Theorem 1.5 implies that $r_1(g) = r_2(g) = \lambda_1(g) = \lambda_2(g) = 0$ if and only if $g : S^3 \sqcup S^3 \rightarrow S^6$ is trivial. Also, $\lambda_1(g) = \lambda_2(g) = 0$ if and only if $g : S^3 \sqcup S^3 \rightarrow S^6$ is unlinked; the “if” part follows from the definitions of λ_1 and λ_2 , and the “only if” part follows from the PL version of Theorem 1.5, see Theorem 1.16.

We use Theorems 1.1, 1.3, 1.5 to prove Theorem 1.11 which is the main result of the paper. First we present two corollaries of Theorem 1.11 showing that the connection between Theorems 1.1, 1.5, 1.3 on one hand and Theorem 1.11 on the other hand is not trivial. The corollaries are proved at the end of this subsection.

For the rest of the text let M_1 and M_2 be some closed connected orientable 3-manifolds.

Corollary 1.7. *Suppose that $H_1(M_1)$ is infinite. Then there is an element $[f] \in E^6(M_1 \sqcup M_2)$ and a non-trivial not unlinked element $[g] \in E^6(S^3 \sqcup S^3)$ such that $[f] \# [g] = [f]$.*

Remark 1.8. If one omits the “ g is not unlinked” part of the statement, the corollary above will trivially follow from Theorem 1.3 (cf. Corollary 1.4).

Corollary 1.9. *There are manifolds M_1, M_2 , an element $[f] \in E^6(M_1 \sqcup M_2)$, and an unlinked element $[g] \in E^6(S^3 \sqcup S^3)$, such that the restrictions of $[f]$ and $[f] \# [g]$ to each connected component are isotopic, but $[f] \neq [f] \# [g]$.*

Remark 1.10. Informally, Corollary 1.4 means that we can sometimes unknot spherical knots by “dragging” them along a knotted manifold M_1 with infinite $H_1(M_1)$. Corollary 1.9 then means that sometimes this procedure is made impossible by the presence of another manifold M_2 linked with M_1 .

For an embedding $f : M_1 \sqcup M_2 \rightarrow S^6$ and $k \in \{1, 2\}$ define

$$W_k : E^6(M_1 \sqcup M_2) \rightarrow H_1(M_k) \quad \text{by the formula} \quad W_k(f) = W(f|_{M_k}).$$

I.e., $W_k(f)$ is the Whitney invariant of the restriction of f to the k -th connected component. The map

$$L_1 \times L_2 : E^6(M_1 \sqcup M_2) \rightarrow H_1(M_1) \times H_1(M_2)$$

is defined below in §1.6. All four W_1, L_1, W_2, L_2 are called (generalized) Whitney invariants.

For brevity we denote

$$WL := W_1 \times L_1 \times W_2 \times L_2$$

for the rest of the text.

For any $[f] \in E^6(M_1 \sqcup M_2)$ let $\text{Stab}_f \subset \widetilde{\mathbb{Z}}^4$ be the subgroup generated by all elements

- $(0, 2L_1 f \cap \alpha, W_1 f \cap \alpha, 0) \in \widetilde{\mathbb{Z}}^4,$
- $(2L_1 f \cap \beta, 2W_1 f \cap \beta, 0, 0) \in \widetilde{\mathbb{Z}}^4,$

- $(2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) \in \widetilde{\mathbb{Z}}^4$,
- $(2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \in \widetilde{\mathbb{Z}}^4$,

where α, β take all values in $H_2(M_1)$ and γ, δ take all values in $H_2(M_2)$, and \cap denotes the cap product.

Theorem 1.11. *For any closed connected orientable 3-manifold M_1 and M_2*

(I) *the map*

$$WL : E^6(M_1 \sqcup M_2) \rightarrow H_1(M_1) \times H_1(M_1) \times H_1(M_2) \times H_1(M_2)$$

is surjective.

(II) *The embedded connected sum action of $E^6(S^3 \sqcup S^3)$ is transitive on each of the preimages of WL .*

(III) *For any $[f] \in E^6(M_1 \sqcup M_2)$ and $[g] \in E^6(S^3 \sqcup S^3)$ the class $[g]$ is in the stabilizer of $[f]$ under the action $\#$ if and only if*

$$(\lambda_1 \times \lambda_2 \times r_1 \times r_2)(g) \in \text{Stab}_f \subset \widetilde{\mathbb{Z}}^4.$$

Parts (II) and (III) of Theorem 1.11 can be restated in terms of description the preimages of WL .

Theorem 1.12. *For any $[f] \in E^6(M_1 \sqcup M_2)$ there is a surjective map*

$$\phi_{[f]} : \widetilde{\mathbb{Z}}^4 \rightarrow WL^{-1}WL(f)$$

such that for any $x, y \in \widetilde{\mathbb{Z}}^4$ we have $\phi_{[f]}(x) = \phi_{[f]}(y)$ if and only if $x - y \in \text{Stab}_f$.

Remark 1.13. In the prequel [Av16] the author proved Theorem 1.11 in the special case of $M_1 = S^1 \times S^2$, $M_2 = S^3$ and only for embeddings $S^1 \times S^2 \sqcup S^3 \rightarrow S^6$ whose restrictions to $S^1 \times S^2$ and S^3 are isotopic to the standard embeddings. Unfortunately, methods used there do not work in the general case. For instance, Corollary 1.9 cannot be deduced from [Av16].

Example 1.14. Suppose that M_1 and M_2 are homology spheres. Then the action $\#$ is transitive and free, and thus gives a 1-1 correspondence between $E^6(M_1 \sqcup M_2)$ and $E^6(S^3 \sqcup S^3)$.

Example 1.15. Suppose that M_1 and M_2 are rational homology spheres (for instance $M_1 = M_2 = \mathbb{R}P^3$). Then each of $|H_1(M_1)| \cdot |H_1(M_2)|$ preimages of WL is in 1-1 correspondence with $E^6(S^3 \sqcup S^3)$.

Proof of Corollary 1.7. Since $H_1(M_1)$ is infinite, there are $\alpha' \in H_1(M_1)$ and $\alpha \in H_2(M_1)$ such that $\alpha' \cap \alpha = 1$.

By part (I) of Theorem 1.11, there is an embedding $f : M_1 \sqcup M_2 \rightarrow S^6$ such that $W_1f = 0$ and $L_1f = \alpha'$.

By Theorem 1.5, there is an embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ such that

$$(\lambda_1 \times \lambda_2 \times r_1 \times r_2)(g) = (0, 2, 0, 0) = (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) \in \text{Stab}_f.$$

Embeddings f and g are as required. Indeed, g is not unlinked, see Remark 1.6, and $[f] = [f]\#[g]$ by part (III) of Theorem 1.11. \square

Proof of Corollary 1.9. Take $M_1 = S^1 \times S^2$ and $M_2 = S^3$. By part (I) of Theorem 1.11, there is an embedding $f : S^1 \times S^2 \sqcup S^3 \rightarrow S^6$ such that $W_1(f) = L_1(f) = [S^1 \times *]$ and $W_2(f) = L_2(f) = 0$.

By Theorem 1.5, there is an embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ such that $(\lambda_1 \times \lambda_2 \times r_1 \times r_2)(g) = (0, 0, 1, 0)$.

Let us prove that f and g are as required. Embedding g is unlinked, see Remark 1.6. By part (III) of Theorem 1.3, we have that the restrictions of $[f]$ and $[f]\#[g]$ to each connected component are isotopic.

It remains to check that $[f] \neq [f]\#[g]$. The group Stab_f is generated by two elements, $(0, 2, 1, 0)$ and $(2, 2, 0, 0)$ of \mathbb{Z}^4 (one can obtain these generators by substituting $\alpha = \beta = [* \times S^2]$ in the definition of Stab_f). Clearly, $(\lambda_1 \times \lambda_2 \times r_1 \times r_2)(g) = (0, 0, 1, 0)$ is not a linear combination of $(0, 2, 1, 0)$ and $(2, 2, 0, 0)$. So, $[f] \neq [f]\#[g]$ by part (III) of Theorem 1.11. \square

1.2. PL version of the main result. For a PL manifold N denote by $E_{PL}^m(N)$ the set of PL isotopy classes of PL embeddings $N \rightarrow S^m$.

In this subsection M_k also denotes the PL manifold obtained by triangulating the smooth manifold M_k . In dimension 3 any PL manifold may be obtained in this way, see for example [Wh61].

The definition of linking coefficients

$$\lambda_1, \lambda_2 : E_{PL}^6(S^3 \sqcup S^3) \rightarrow \mathbb{Z},$$

of Whitney invariants

$$WL : E_{PL}^6(M_1 \sqcup M_2) \rightarrow H_1(M_1) \times H_1(M_1) \times H_1(M_2) \times H_1(M_2),$$

and of the (componentwise) embedded connected sum $\#$ carries over from the smooth category without any changes.

The set $E_{PL}^6(S^3 \sqcup S^3)$ is still a group with $\#$ being the group operation.

Theorem 1.16 (A. Haefliger, [Ha62a]). *The map $\lambda_1 \times \lambda_2 : E_{PL}^6(S^3 \sqcup S^3) \rightarrow \mathbb{Z}^2$ is a monomorphism and its image is $\{(a, b) \in \mathbb{Z}^2 \mid a \equiv b \pmod{2}\}$.*

For any $[f] \in E_{PL}^6(M_1 \sqcup M_2)$ let $\text{Stab}_{PL,f} \subset \mathbb{Z}^2$ be the subgroup generated by all elements

- $(0, 2L_1f \cap \alpha)$,
- $(2L_1f \cap \beta, 2W_1f \cap \beta)$,
- $(2L_2f \cap \gamma, 0)$,
- $(2W_2f \cap \delta, 2L_2f \cap \delta)$,

where α, β take all values in $H_2(M_1)$ and γ, δ take all values in $H_2(M_2)$.

Remark 1.17. In the definition of $\text{Stab}_{PL,f}$ one can replace $(0, 2L_1f \cap \alpha)$ and $(2L_2f \cap \gamma, 0)$ by $(0, 2\text{div}(L_1f))$ and $(2\text{div}(L_2f), 0)$, respectively. We do not know of any further simplifications.

Theorem 1.18. *For any closed connected orientable PL 3-manifold M_1 and M_2*

(I) *the map*

$$WL : E_{PL}^6(M_1 \sqcup M_2) \rightarrow H_1(M_1) \times H_1(M_1) \times H_1(M_2) \times H_1(M_2)$$

is surjective.

(II) *The embedded connected sum action of $E_{PL}^6(S^3 \sqcup S^3)$ is transitive on each of the preimages of WL .*

(III) *For any $[f] \in E_{PL}^6(M_1 \sqcup M_2)$ and $[g] \in E_{PL}^6(S^3 \sqcup S^3)$ the class $[g]$ is in the stabilizer of $[f]$ under the action $\#$ if and only if $(\lambda_1 \times \lambda_2)(g) \in \text{Stab}_{PL,f} \subset \mathbb{Z}^2$.*

1.3. A very brief survey on embeddings classification. According to E. C. Zeeman ([Ze93], [MAa]), three major classical problems of topology are the following.

- *Homeomorphism Problem:* Classify n -manifolds.
- *Embedding Problem:* Find the least dimension m such that given space admits an embedding into m -dimensional Euclidean space \mathbb{R}^m .
- *Knotting Problem:* Classify embeddings of a given space into another given space up to isotopy.

This paper is on a special case of the third problem.

Let us start with the known results on the sets $E^m(S^n)$ and $E_{PL}^m(S^n)$. The set $E^3(S^1)$ (or $E_{PL}^3(S^1)$) is studied in the classical knot theory which produced a lot of beautiful results in the last 200 years. However, relatively early it was understood that a complete classification of $E^3(S^1)$ is probably unachievable. In general, there is no known complete classification of $E^m(S^n)$ for $m = n + 2 \geq 3$.

The situation is much better when $m \geq n + 3$ (codimension at least 3 case). So, until the end of this subsection we assume that $m \geq n + 3$.

The following two theorems establish that there are no knots in case when the codimension $m - n$ is large enough. Somewhat surprisingly, the precise meaning of “large enough” is different in the smooth and PL categories.

Theorem (E. C. Zeeman, [Ze63, Theorem 2]). $|E_{PL}^m(S^n)| = 1$.

Theorem (A. Haefliger). *If $2m \geq 3n + 4$ then $|E^m(S^n)| = 1$.*

As it was said earlier, the sets $E^m(S^n)$ and $E_{PL}^m(S^n)$ have a group structure given by the embedded connected sum operation. The following is a generalisation of Theorem 1.1.

Theorem (A. Haefliger). *$E^{3k}(S^{2k-1}) \cong \mathbb{Z}$ for $k > 0$ even and $E^{3k}(S^{2k-1}) \cong \mathbb{Z}_2$ for $k > 1$ odd.*

There is a special embedding $S^{2k-1} \rightarrow S^{3k}$, also called the *Haefliger trefoil knot* (see [Ha62b]), which is a generator of $E^{3k}(S^{2k-1}) \cong \mathbb{Z}$ for even k . It is not known, however, if the Haefliger trefoil knot is the generator of $E^{3k}(S^{2k-1}) \cong \mathbb{Z}_2$ for odd k , see [MAB].

Let us now mention a few results on the knotting of manifolds different from spheres.

For any n -connected PL m -manifold N (recall that $m \geq n + 3$) the set $E_{PL}^{2m-n}(N)$ was classified by J. Vrabec in [Vr77].

For any smooth connected 4-manifold N the set $E^7(N)$ was classified only recently and only up to the embedded connected sum action of $E^7(S^4)$ by D. Crowley and A. Skopenkov in [CS16a]. In the sequel [CS16b] the authors strengthened this result. In the special case $H_1(N) = 0$ a complete classification of $E^7(N)$ was obtained much earlier by J. Bo  chat and A. Haefliger in [BH70]. See also [Bo71] for the generalisation to the case of $E^{6k+1}(N)$, where N is $4k$ -dimensional.

Finally, let us get back to links, i.e., isotopy classes of embeddings of manifolds with several connected components. Denote by $S_{(k)}^n$ the disjoint union of k copies of S^n .

Theorem (A. Haefliger, [Ha66]). *There is an isomorphism*

$$E^m(S_{(k)}^n) \rightarrow E_{PL}^m(S_{(k)}^n) \oplus \bigoplus_{i=1}^k E^m(S^n).$$

Composition of the isomorphism with the projection to $E_{PL}^m(S_{(k)}^n)$ is the forgetful map. Composition of the isomorphism with the projection to the i -th summand of $\bigoplus E^m(S^n)$ is the isotopy class of the i -th connected component.

In other words, in codimension at least 3 smooth and PL links of spheres differ only by smooth knotting of each connected component.

For $1 \leq i, j \leq k$, $i \neq j$, let

$$\lambda_{ij} : E_{PL}^m(S_{(k)}^n) \rightarrow \pi_n(S^{m-n-1})$$

be the (generalized) *linking coefficient* of the i -th and the j -th connected components, i.e., the homotopy class of i -th component in the compliment to the j -th component. The map λ_{ij} is analogously defined in the smooth category (in the special case $k = 2$, $n = 3$, $m = 6$, we denote λ_{12} and λ_{21} simply as λ_1 and λ_2 , respectively, throughout the rest of the paper).

Theorem (A. Haefliger, [Ha66]). *The collection of pairwise linking coefficients is bijective for $2m \geq 3n + 4$ and $E_{PL}^m(S_{(k)}^n)$.*

Theorem (A. Haefliger, [Ha62a]). *When $k \geq 2$, $k \neq 3, 7$ the homomorphism*

$$\lambda_{12} \oplus \lambda_{21} : E_{PL}^{3k}(S^{2k-1} \sqcup S^{2k-1}) \rightarrow \pi_{2k-1}(S^k) \oplus \pi_{2k-1}(S^k)$$

is injective and its image is $\{(a, b) : \Sigma a = \Sigma b\}$.

Combining this theorem ($k = 2$) with some of the other theorems above one gets Theorem 1.5, i.e., a classification of $E^6(S^3 \sqcup S^3)$.

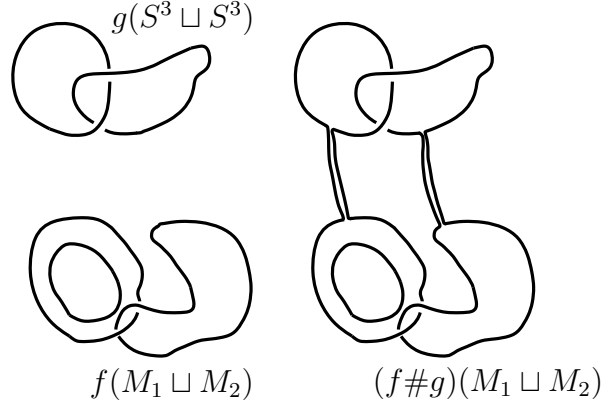
All the results we mentioned so far were either

- in the metastable range $2m \geq 3n + 4$,
- or on links of (homology) spheres,
- or on connected manifolds.

Therefore, Theorem 1.11 is the first embeddings classification result (that we are aware of) which falls into none of those three categories.

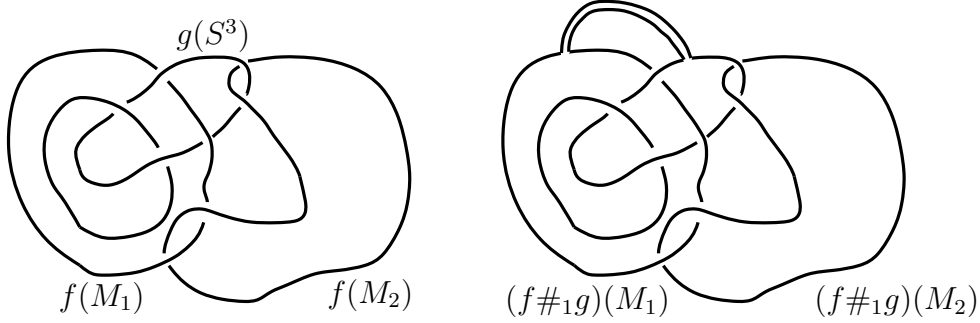
1.4. Definition of embedded connected sum $\#$. Let $f : M_1 \rightarrow S^6$ and $g : S^3 \rightarrow S^6$ be embeddings. Take representatives $f' \in [f]$ and $g' \in [g]$ such that the images of f' and g' lie in disjoint balls. Connect the images of f' and g' by a thin tube along an arc. The isotopy class of the obtained embedding is called an *embedded connected sum* of f and g and is denoted by $[f] \# [g]$. The independence on the choice of the representatives, the arc, and the tube follows by an argument analogous to [Sk15, Standardization Lemma, case $(p, q, m) = (0, 3, 6)$].

For embeddings $f : M_1 \sqcup M_2 \rightarrow S^6$ and $g : S^3 \sqcup S^3 \rightarrow S^6$ their *component-wise* embedded connected sum is defined analogously and is also denoted by $[f] \# [g]$, see Fig.1.

FIGURE 1. The componentwise embedded connected sum $\#$.

The described operation $\#$ defines a group structure on $E^6(S^3)$ (or $E^6(S^3 \sqcup S^3)$) and an action of $E^6(S^3)$ (or $E^6(S^3 \sqcup S^3)$) on $E^6(M_1)$ (or $E^6(M_1 \sqcup M_2)$).

1.5. Definition of linked embedded connected sum $\#_1, \#_2$. Let $f : M_1 \sqcup M_2 \rightarrow S^6$ and $g : S^3 \rightarrow S^6$ be embeddings with disjoint images. For $k \in \{1, 2\}$ connect $f(M_k)$ with $g(S^3)$ by a thin tube along an arc. Denote the obtained embedding $M_1 \sqcup M_2 \rightarrow S^6$ by $f \#_k g$. It is called a *linked embedded connect sum* of f and g . Clearly, the embedding $f \#_k g$ depends on the choice of the arc and the tube, but we drop them from the notation. See Fig.2.

FIGURE 2. The linked embedded connected sum $\#_1$.

For the fixed embeddings f and g the isotopy class $[f \#_k g]$ is well defined, i.e., it does not depend on the choice of the arc or the tube. This can be proved analogously to [Sk15, Standardization Lemma, case $(p, q, m) = (0, 3, 6)$] (the independence on the choice of the arc also easily follows from the fact that the images of f and g have codimension greater than 2).

1.6. Definition of the Whitney invariants W and L_k . Let N be a closed connected orientable 3-manifold. Our definition of the Whitney invariant $W : E^6(N) \rightarrow H_1(N)$ is equivalent to the one given in [Sk08a].

Let $f, f' : N \rightarrow S^6$ be embeddings. Consider a general position homotopy $F : N \times I \rightarrow S^6 \times I$ between f and f' . The Whitney invariant of the pair (f, f') is the homology class

$$W(f, f') := [\{x \in N \times I : |F^{-1}Fx| \geq 2\}] \in H_1(N \times I) = H_1(N)$$

which can be defined as in [Sk08b].

To define W for a single embedding (as opposed to a pair (f, f') of embeddings) we need to choose some “base embedding”. Manifold N is orientable, so it embeds into S^5 , see [Hi61]. Let $f_N^0 : N \rightarrow S^6$ be an embedding with the image in $S^5 \subset S^6$. For any $f : N \rightarrow S^6$ denote

$$W(f) := W(f_N^0, f).$$

We choose $f_{M_1}^0$ and $f_{M_2}^0$ so that their images lie in disjoint 6-balls. Define

$$f^0 : M_1 \sqcup M_2 \rightarrow S^6 \quad \text{by the formula} \quad f^0 = f_{M_1}^0 \sqcup f_{M_2}^0.$$

Recall that for $k \in \{1, 2\}$ and for an embedding $f : M_1 \sqcup M_2 \rightarrow S^6$ we earlier defined

$$W_k : E^6(M_1 \sqcup M_2) \rightarrow H_1(M_k) \quad \text{by the formula} \quad W_k(f) = W(f|_{M_k}).$$

Let us now define L_1 and L_2 . Let $f, f' : M_1 \sqcup M_2 \rightarrow S^6$ be embeddings. Consider a general position homotopy $F : (M_1 \sqcup M_2) \times I \rightarrow S^6 \times I$ between f and f' . The Whitney invariants L_1 and L_2 of the pair (f, f') are the homology classes

$$\begin{aligned} L_1(f, f') &:= [(F|_{M_1 \times I})^{-1}(F(M_1 \times I) \cap F(M_2 \times I))] \in H_1(M_1), \\ L_2(f, f') &:= [(F|_{M_2 \times I})^{-1}(F(M_1 \times I) \cap F(M_2 \times I))] \in H_1(M_2). \end{aligned}$$

For $k \in \{1, 2\}$ and any $f : M_1 \sqcup M_2 \rightarrow S^6$ denote

$$L_k(f) := L_k(f^0, f) \in H_1(M_k).$$

1.7. Proof of part (I) of Theorem 1.11. The following claim is essentially proved (but not explicitly stated) in [Sk08a, “Construction of an arbitrary embedding $N \rightarrow \mathbb{R}^6$ from a fixed embedding $g : N \rightarrow \mathbb{R}^5$ ”]. For the readers convenience we present (a very similar) proof here. In the proof an later in the text we use the standard notation $V_{m,n}$ for the Stiefel manifold of n -frames in \mathbb{R}^m . All the framings (resp. frames) in the text are *normal* framings (resp. frames) compatible with orientation (in the case of framings).

Claim 1.19. *Let $f : M_1 \sqcup M_2 \rightarrow S^6$ be an embedding and $a \in H_1(M_1)$ a homology class. Then there is an embedding $g : D^4 \rightarrow S^6$ such that*

- $g(S^3) \cap \text{Im}(f) = \emptyset$,
- $g(D^4) \cap f(M_2) = \emptyset$,
- $[(f|_{M_1})^{-1}g(D^4)] = a$.

Proof. Represent a by an oriented circle in M_1 and denote the circle by the same letter. Consider a normal framing α of $f(a)$ in $f(M_1)$. Extend it to a normal framing α, β of $f(a)$ in S^6 , where β is normal to $f(M_1)$. The extension exists because $f(a)$ is unknotted in S^6 and so the obstruction to the existence of the extension is in $\pi_1(V_{5,2}) = 0$.

By general position there is a 2-disk Δ in S^6 such that

- $\partial\Delta = f(a)$,
- $\text{Int}\Delta \cap f(M_1 \sqcup M_2) = \emptyset$,

- the first vectors of β “looks” inside of Δ .

Denote by β' the normal 2-frame of $f(a)$ made out of the last two vectors of β . Extend β' to a normal 2-frame of Δ . The extension exists because the obstruction to its existence is in $\pi_1(V_{4,2}) = 0$. The vectors of β' on Δ plus the vectors of β on $\partial\Delta = f(a)$ give us an embedding $g : D^4 \rightarrow D^6$ which is as required. \square

Proof of part (I) of Theorem 1.11. We need to prove that WL is surjective.

Take any element $a' \in H_1(M_1)$ and any embedding $f : M_1 \sqcup M_2 \rightarrow S^6$. Denote $a := a' - W_1(f)$. Let $g : D^4 \rightarrow S^6$ be an embedding given by Claim 1.19.

Consider the embedding $f' := f \#_1(g|_{S^3})$. There is a homotopy between f' and f contracting $g(S^3)$ along the disk $g(D^4)$. By the definition of the Whitney invariants and by the construction of g , we have $W_1(f') = W_1(f) + a = a'$, and $W_2(f') = W_2(f)$, $L_1(f') = L_1(f)$, $L_2(f') = L_2(f)$. So, we can change the value of the Whitney invariant W_1 of an embedding to any desired value a' without changing the other three Whitney invariants.

Similarly to the previous paragraph (take $f'' := f \#_2(g|_{S^3})$ instead of f') we can change the value of the Whitney invariant L_1 of an embedding to any desired value a' without changing the other three Whitney invariants.

Similarly to previous two paragraphs we can also change W_2 and L_2 in the same manner. So, WL is surjective, because there exists at least one embedding $M_1 \sqcup M_2 \rightarrow S^6$ (for instance take f^0). \square

1.8. Definition of the linking coefficients λ_1 and λ_2 and their relation to the Haefliger invariant r . Let $g : S_1^3 \sqcup S_2^3 \rightarrow S^6$ be an embedding, where S_1^3 and S_2^3 are two copies of S^3 . Choose an oriented disk $D_g^3 \subset S^6$ intersecting $g(S_2^3)$ transversally at a single point of positive sign. Identify $H_2(S^6 \setminus gS_2^3)$ with \mathbb{Z} by identifying $[\partial D_g^3] \in H_2(S^6 \setminus gS_2^3)$ with $1 \in \mathbb{Z}$. Identify $H_2(S^2)$ with \mathbb{Z} by choosing an orientation of S^2 . Choose a homotopy equivalence $h : S^6 \setminus gS_2^3 \rightarrow S^2$ which induces the identity isomorphism in H_2 . Define the first linking coefficient by the formula

$$\lambda_1(g) := [hg|_{S_1^3}] \in \pi_3(S^2) = \mathbb{Z},$$

where identification $\pi_3(S^2) = \mathbb{Z}$ identifies the homotopy class of the Hopf map with 1. All the orientation preserving homotopy equivalences $S^2 \rightarrow S^2$ are homotopic to each other, so λ_1 is well-defined.

The definition of the second linking coefficient λ_2 is analogous and is obtained by the exchange of the components,

$$\lambda_2(g) := \lambda_1(g'),$$

where $g' : S_1^3 \sqcup S_2^3 \rightarrow S^6$ is such that $g'|_{S_2^3} = g|_{S_1^3}$ and $g'|_{S_1^3} = g|_{S_2^3}$.

Let $A, B : S^3 \rightarrow S^6$ be embeddings with disjoint images. For brevity denote

$$\lambda(A, B) := \lambda_1(A \sqcup B).$$

Informally, $\lambda(A, B)$ is the homotopy class of A in the complement to $B(S^3)$.

The following lemma easily follows from the definition of λ .

Lemma 1.20. *Let $A, B, C : S^3 \rightarrow S^6$ be embeddings with pairwise disjoint images. Then*

$$\lambda(A \# B, C) = \lambda(A, C) + \lambda(A, B).$$

Remark 1.21. Note that $\lambda(A, B \# C)$ is not necessarily equal to $\lambda(A, B) + \lambda(A, C)$ even if the images of B and C lie in pairwise disjoint 6-balls. As an example one can take Borromean rings $A, B, C : S^3 \rightarrow S^6$. Then $A \sqcup B \# C : S^3 \sqcup S^3 \rightarrow S^6$ is the Whitehead link with $\lambda(A, B \# C) = 2 \neq 0 + 0 = \lambda(A, B) + \lambda(A, C)$, see [Sk15, Lemma 2.18].

For the proof of the following lemma see [Sk15, Lemma 2.16].

Lemma 1.22. *Let $A, B : S^3 \rightarrow S^6$ be embeddings with disjoint images. Then*

$$r(A \# B) = r(A) + r(B) + \frac{\lambda(A, B) + \lambda(B, A)}{2}.$$

In particular, $r(A \# B) = r(A) + r(B)$ if $A(S^3)$ and $B(S^3)$ lie in disjoint 6-balls.

Remark 1.23. The number $\frac{\lambda(A, B) + \lambda(B, A)}{2}$ is integer by Haefliger Theorem 1.5.

1.9. Proof of “PL” Theorem 1.18 modulo “smooth” Theorem 1.11. For a piecewise smooth (PS) manifold N denote by $E_{PS}^m(N)$ the set of PS isotopy classes of PS embeddings $N \rightarrow S^m$. The forgetful map $E_{PL}^m(N) \rightarrow E_{PS}^m(N)$ is a bijection, see [Ha67, §2.2]. Therefore, Theorem 1.18 can be restated in the PS category without any changes. For our convenience we shall prove the PS version of Theorem 1.18.

Let

$$\text{Fg} : E^6(M_1 \sqcup M_2) \rightarrow E_{PS}^6(M_1 \sqcup M_2)$$

be the *forgetful* map.

Lemma 1.24. *The forgetful map Fg has the following properties.*

- (1) *Fg preserves the invariants λ_1 , λ_2 , and WL .*
- (2) *Fg commutes with $\#$, i.e., $\text{Fg}([f] \# [g]) = \text{Fg}([f]) \# \text{Fg}([g])$ for any $[f] \in E^6(M_1 \sqcup M_2)$ and $[g] \in E_{PL}^6(S^3 \sqcup S^3)$.*
- (3) *Fg is surjective.*
- (4) *Suppose that $\text{Fg}([f']) = \text{Fg}([f])$ for some $[f], [f'] \in E^6(M_1 \sqcup M_2)$. Then there is $[g] \in E^6(S^3 \sqcup S^3)$ such that $[f'] = [f] \# [g]$ and that $[g]$ is unlinked, i.e., $\lambda_1(g) = \lambda_2(g) = 0$.*

Proof. (1), (2) follow by the definitions of λ_1 , λ_2 , WL , and $\#$.

Let us prove (3). The obstruction to smoothing any PS embedding $M_1 \sqcup M_2 \rightarrow S^6$ lies in groups $H^{i+1}(M_1 \sqcup M_2; E^{i+3}(S^i))$ for $i = 0, 1, 2$, see [Bo71, First paragraph of introduction], [Hu72, Proof of Lemma 7]. Since $E^3(S^0) = E^4(S^1) = E^5(S^2) = 0$, the obstruction vanishes.

It remains to prove (4). Let $F : (M_1 \sqcup M_2) \times I \rightarrow S^6 \times I$ be a PS isotopy between f and f' . The only obstruction to smoothing F is some cohomology class $a \in H^4((M_1 \sqcup M_2) \times I; E^6(S^3)) \cong E^6(S^3) \oplus E^6(S^3)$. Choose an unlinked embedding $g : S^3 \sqcup S^3 \rightarrow S^6 \times 0$ whose image is in a 6-ball disjoint with the image of F_0 and such that $r_1(g) \oplus r_2(g) = a$. A PS embedding $G : D^4 \sqcup D^4 \rightarrow S^6 \times I$ is obtained from g by coning over two generic points. Then $F \# G$ is a PS concordance between $[f] \# [g]$ and $[f']$. By construction, $F \# G$ can be smoothed, therefore $[f'] = [f] \# [g]$. Cf. [Sk08a, An alternative definition of the Kreck invariant]. \square

Proof of Theorem 1.18. Part (I) follows from Part (I) of Theorem 1.11 by (1) and (3). Part (II) follows from Part (II) of Theorem 1.11 by (1), (2), and (3). Part (III) follows from Part (II) of Theorem 1.11 by (1), (2), (3), and (4). \square

2. PROOF OF THE MAIN THEOREM MODULO LEMMAS.

2.1. Plan of the proof. In this section we prove the main theorem modulo Surjectivity Lemma 2.3, Bijectivity Lemma 2.7, Preimage Lemma 2.9, Calculation Lemma 2.11, Linking Lemma 2.12, and Claim 2.8. All of these statements are proved later in the corresponding sections.

The plan of the proof is explained by the diagram in Fig.3. In this subsection we only give informal explanations. All the new objects and statements mentioned here or in the diagram are rigorously defined or stated later in this section.

We represent M_1 as the result of cutting several solid tori from S^3 and then pasting them back together by the diffeomorphism exchanging parallels with meridians. By \widehat{M}_1 we denote the compliment in S^3 to the solid tori, i.e., what is left of S^3 after cutting the tori and before pasting them back. The definition of \widehat{M}_2 is analogous.

By $\widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ we denote the set of fixed on the boundary isotopy classes of proper embeddings $\widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$. Given a representative of an element of $\widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ we can extend it in two different “standard” ways to either an embedding $S^3 \sqcup S^3 \rightarrow S^6$ or an embedding $M_1 \sqcup M_2 \rightarrow S^6$. These extensions define the maps σ_R and σ in the diagram.

It turns out that the map σ (and σ_R) is surjective, see the Surjectivity Lemma 2.3. I.e., any embedding $M_1 \sqcup M_2 \rightarrow S^6$ is isotopic to a so-called “standardized” embedding which is “standard” on the solid tori and which maps $\widehat{M}_1 \sqcup \widehat{M}_2$ to D_+^6 . The proof of Surjectivity Lemma 2.3 essentially repeats the proof of the first part of the Standardization Lemma in [Sk15] (which is stated in slightly less general case than we require).

Two isotopic “standardized” embeddings are not necessarily isotopic through “standardized” embeddings. This means that the map σ is not bijective (and that the second part of the Standardization Lemma of [Sk15] fails in the dimensions we are working in). By studying the geometric obstruction to the “standardization” of an isotopy between two “standardized” embeddings we prove the Preimage Lemma 2.9.

The set $E^6(S^3 \sqcup S^3)$ is known and the maps σ and σ_R are surjective. Therefore we can classify the unknown set $E^6(M_1 \sqcup M_2)$ by describing the (not well-defined) “composition” $\sigma_R \circ \sigma^{-1}$. This task is accomplished by the Bijectivity, Preimage, and Calculation Lemmas 2.7, 2.9, and 2.11.

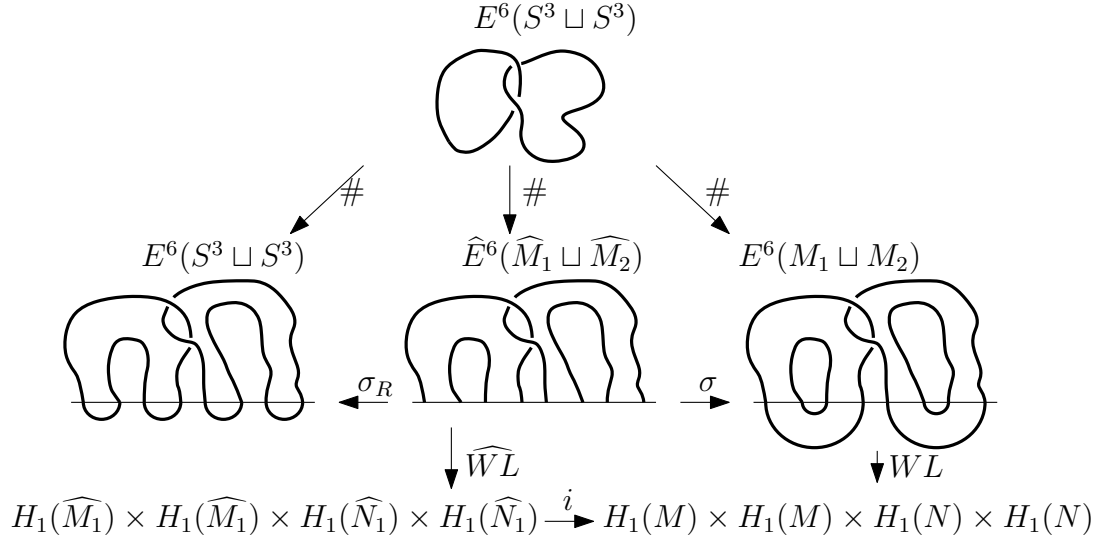


FIGURE 3. The diagram.

2.2. Definitions of $T_n, P, \widehat{M}_k, m, R$. In this subsection we represent manifolds M_1 and M_2 as results of a surgery of S^3 on several embedded circles.

For any $n > 0$ let

$$T_n := \underbrace{S^1 \times D^2 \sqcup \dots \sqcup S^1 \times D^2}_n$$

be the disjoint union of n copies of $S^1 \times D^2$.

Let

$$R : S^1 \times S^1 \rightarrow S^1 \times S^1$$

be the diffeomorphism exchanging the parallel with the meridian.

By [PS97, end of §12, beginning of §14] for each $k \in \{1, 2\}$ there are $m_k > 0$ and an embedding $P_k : T_{m_k} \rightarrow S^3$ such that

- the restriction of P_k to each of m_k connected components of T_{m_k} is isotopic to the standard embedding $S^1 \times D^2 \rightarrow S^3$;
- if we denote

$$\widehat{M}_k := \text{the closure of } S^3 \setminus P_k(T_{m_k})$$

then

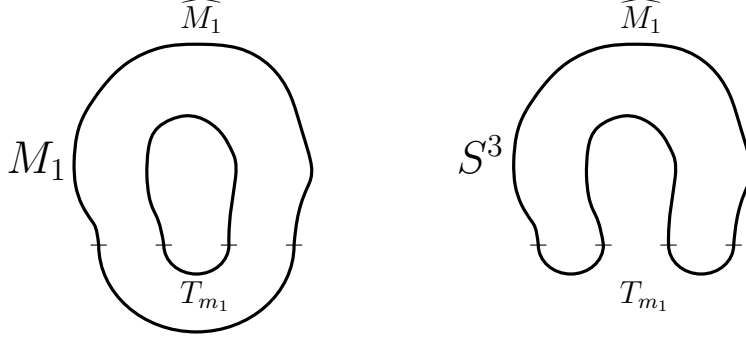
$$M_k \cong \widehat{M}_k \bigcup_{P_k(x)=R(x), x \in \partial T_{m_k}} T_{m_k},$$

(where “ \cong ” is a diffeomorphism).

For the rest of the text and for each $k \in \{1, 2\}$ we replace M_k with

$$\widehat{M}_k \bigcup_{P_k(x)=R(x), x \in \partial T_{m_k}} T_{m_k},$$

see Fig.4.

FIGURE 4. Manifolds M_1 on the left and S^3 on the right.

Until the end of the text $k \in \{1, 2\}$ and $1 \leq i \leq m_k$. I.e., all the statements involving k and/or i are given *for all* $k \in \{1, 2\}$ and $1 \leq i \leq m_k$, unless specifically said otherwise.

2.3. Definitions of $P_{k,i}$, $\mathfrak{p}_{k,i}$, h . Denote by $P_{k,i}$ the restriction of P_k to the i -th connected component.

Fix an orientation of $S^1 \times D^2$. Consider the meridian $\mathfrak{m} := * \times S^1 \subset S^1 \times D^2$ with some orientation. Construct a normal framing of \mathfrak{m} in the following way. The first vector of the framing “looks” inside the full-torus $S^1 \times D^2$, the second vector of the framing is then determined uniquely by the compatibility with orientation. Denote the obtained framed circle by the same letter \mathfrak{m} .

Define framed circles $\mathfrak{p}_{k,i} \subset S^3$ by the formula

$$\mathfrak{p}_{k,i} := P_{k,i} R\mathfrak{m} \subset S^3.$$

Let $a \subset S^3$ be any framed 1-submanifold. Shift a slightly along the first vector of its framing and denote the obtained submanifold by a' . The *Hopf invariant* $h(a)$ of a is defined by the formula

$$h(a) := \text{lk}(a, a') \in \mathbb{Z}.$$

The following claim easily follows from the definition of $\mathfrak{p}_{k,i}$.

Claim 2.1. *For any $k \in \{1, 2\}$ and $1 \leq i \leq m_k$ we have $h(\mathfrak{p}_{k,i}) = 0$.*

2.4. Definition of the set $\widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$. Denote by D_+^6 and D_-^6 the northern and the southern hemispheres of S^6 , respectively (the exact choice of the “north” and “south” poles is not important).

Let

$$s_k : \underbrace{D^2 \times D^4 \sqcup \dots \sqcup D^2 \times D^4}_{m_k} \rightarrow D_-^6$$

be an embedding such that

- its restriction to each $* \times D^4$ is isotopic to the standard proper embedding $D^4 \rightarrow D_-^6$,
- there are pairwise disjoint 6-balls $B_{k,i} \subset D_-^6$ such that the s_k -image of the i -th connected component lie in $B_{k,i}$.

We additionally demand that for every $1 \leq i \leq m_1$, $1 \leq j \leq m_2$ the balls $B_{1,i}$ and $B_{2,j}$ are disjoint.

Denote by $B_{k,i}^\square$ some tubular neighbourhood of $s_{k,i}(D^2 \times D^4)$ in $B_{k,i}$ modulo $s_{k,i}(D^2 \times S^3)$. Note that $B_{k,i}^\square$ is a manifold with “corners” diffeomorphic to $D^2 \times D^4$.

We consider $S^1 \times D^2$ as a submanifold of $D^2 \times D^4$ where the inclusion $S^1 \times D^2 \subset D^2 \times D^4$ is given by the obvious inclusions $S^1 = \partial D^2 \subset D^2$ and $D^2 \subset D^4$.

Denote by $\widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ the set of isotopy classes fixed on the boundary, of proper embeddings $\widehat{f} : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$ such that

$$\widehat{f} \circ P_k|_{\partial T_{m_k}} = s_k|_{\partial T_{m_k}}$$

for each $k \in \{1, 2\}$.

2.5. Definition of operations σ , σ_R and the action $\#$. For an embedding $\widehat{f} : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$ such that $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ define

$$\sigma(\widehat{f}) : M_1 \sqcup M_2 \rightarrow S^6 \quad \text{by} \quad \sigma(\widehat{f})(x) := \begin{cases} \widehat{f}(x) & \text{if } x \in \widehat{M}_1 \sqcup \widehat{M}_2 \\ s_k(x) & \text{if } x \in (M_k \setminus \widehat{M}_k) = T_{m_k} \end{cases},$$

see Fig.5.

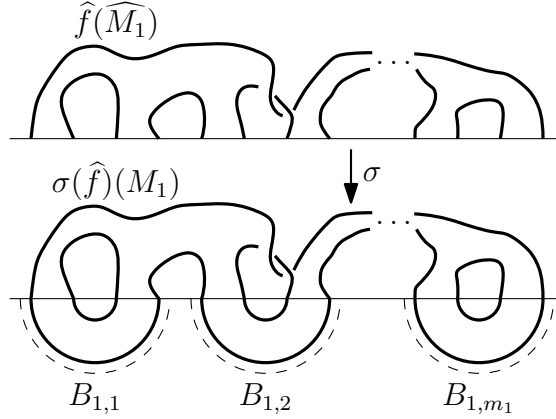


FIGURE 5. Operation σ (only M_1 is shown).

Denote by $s_{k,i}$ the restrictions of s_k to the i -th connected component.

Let $A_R : D_-^6 \rightarrow D_-^6$ be an orientation preserving diffeomorphism such that

- $A_R(B_{k,i}) = B_{k,i}$,
- $A_R \circ s_{k,i}|_{S^1 \times S^1} = s_{k,i}|_{S^1 \times S^1} \circ R$.

Such a diffeomorphism exists because all embeddings $S^1 \times S^1 \rightarrow \partial D_-^6$ are isotopic and because smooth isotopies are ambient, see [Hu70, Theorem 2.1].

Denote

$$s_{R,k} := A_R \circ s_k \quad \text{and} \quad s_{R,k,i} := A_R \circ s_{k,i}.$$

For an embedding $\widehat{f} : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$ such that $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ define

$$\sigma_R(\widehat{f}) : M_1 \sqcup M_2 \rightarrow S^6 \quad \text{by} \quad \sigma_R(\widehat{f})(x) := \begin{cases} \widehat{f}(x) & \text{if } x \in \widehat{M}_1 \sqcup \widehat{M}_2 \\ s_{R,k}(x) & \text{if } x \in (M_k \setminus \widehat{M}_k) = T_{m_k} \end{cases},$$

see Fig.6.

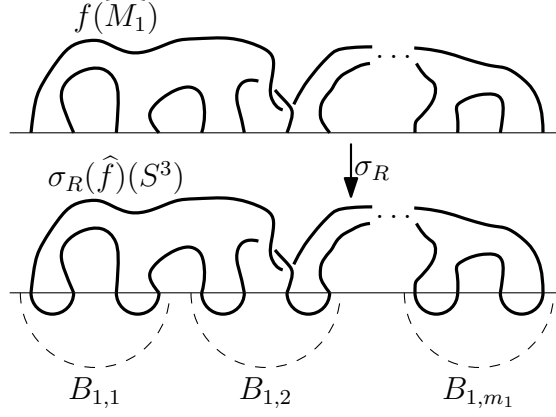


FIGURE 6. Operation σ_R (only M_1 is shown).

Clearly, if $[\hat{f}] = [\hat{f}']$ for some other embedding \hat{f}' , then $[\sigma(\hat{f})] = [\sigma(\hat{f}')]$ and $[\sigma_R(\hat{f})] = [\sigma_R(\hat{f}')]$. Therefore σ and σ_R induce well-defined maps

$$E^6(S^3 \sqcup S^3) \xleftarrow{\sigma_R} \hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2) \xrightarrow{\sigma} E^6(M_1 \sqcup M_2),$$

which we denote by the same letters.

Note that in our notation $\sigma(\hat{f})$ is an *embedding* while $\sigma([\hat{f}])$ is an *isotopy class*.

The group $E^6(S^3 \sqcup S^3)$ acts on each of the sets $E^6(S^3 \sqcup S^3)$, $\hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$, and $E^6(M_1 \sqcup M_2)$ via the component-wise connected sum $\#$. The action on $\hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ is defined analogously to the action on $E^6(S^3 \sqcup S^3)$ or $E^6(M_1 \sqcup M_2)$.

The following claim easily follows from the definitions of σ , σ_R , and the embedded connected sum action $\#$.

Claim 2.2 ($\#$ -commutativity). *The embedded connected sum action $\#$ commutes with σ and σ_R . I.e., for any isotopy classes $[\hat{f}] \in \hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ and $[g] \in E^6(S^3 \sqcup S^3)$ we have $\sigma([\hat{f}] \# [g]) = \sigma([\hat{f}]) \# [g]$ and $\sigma_R([\hat{f}] \# [g]) = \sigma_R([\hat{f}]) \# [g]$.*

Lemma 2.3 (Surjectivity). *Maps σ and σ_R are surjective.*

2.6. Definition of the Whitney invariants \widehat{W}_k , \widehat{L}_k of proper embeddings.

The definition of

$$\widehat{W}_k, \widehat{L}_k : \hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2) \rightarrow H_1(\widehat{M}_k)$$

is analogous to the definition of

$$W_k, L_k : E^6(M_1 \sqcup M_2) \rightarrow H_1(M_k).$$

One needs only to replace “homotopy” by “homotopy relative to the boundary” and define a “base embedding” $\hat{f}^0 : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$. To do the latter we choose some $[\hat{f}^0] \in \hat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ such that $\sigma([\hat{f}^0]) = [f^0]$. The existence of such $[\hat{f}^0]$ is guaranteed by Surjectivity Lemma 2.3.

The following claim easily follows from the definition of \widehat{L}_k .

Claim 2.4. Take any $k \in \{1, 2\}$ and $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$. Let $\Delta_k \subset D_+^6$ be a proper submanifold “with corners”, $\partial\Delta_k = \widehat{f}(\widehat{M}_k) \cup (\partial\Delta_k \cap \partial D_+^6)$. Suppose that Δ_k is disjoint with $\widehat{f}(\partial\widehat{M}_{3-k}) \subset \partial D_+^6$. Then

$$\widehat{L}_{3-k}(\widehat{f}) = [(\widehat{f}^{-1})\Delta_k] \in H_1(\widehat{M}_{3-k}).$$

For brevity, denote

$$\widehat{WL} := \widehat{W}_1 \times \widehat{L}_1 \times \widehat{W}_2 \times \widehat{L}_2.$$

The map

$$H_1(\widehat{M}_1) \times H_1(\widehat{M}_1) \times H_1(\widehat{M}_2) \times H_1(\widehat{M}_2) \rightarrow H_1(M_1) \times H_1(M_1) \times H_1(M_2) \times H_1(M_2)$$

in the diagram is induced by the inclusions $\widehat{M}_1 \subset M_1$ and $\widehat{M}_2 \subset M_2$.

Our choice of the “base element” $[\widehat{f}^0] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ implies the following two claims.

Claim 2.5. For any $k \in \{1, 2\}$, $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ and $[f] := \sigma([\widehat{f}])$ the homology classes $\widehat{W}_k(\widehat{f})$ and $W_k(f)$ can be represented by the same 1-submanifold in \widehat{M}_k . Likewise, the homology classes $\widehat{L}_k(\widehat{f})$ and $L_k(f)$ can be represented by the same 1-submanifold in \widehat{M}_k .

Claim 2.6. The square in the diagram above commutes.

2.7. Proof of part (II) of Theorem 1.11.

Lemma 2.7 (Bijectivity). For any $x \in H_1(\widehat{M}_1) \times H_1(\widehat{M}_1) \times H_1(\widehat{M}_2) \times H_1(\widehat{M}_2)$ the restriction $\sigma_R|_{\widehat{WL}^{-1}(x)}$ is a bijection.

Claim 2.8. Let $[f], [f'] \in E^6(M_1 \sqcup M_2)$ be isotopy classes such that $WL(f) = WL(f')$. Then there are isotopy classes $[\widehat{f}], [\widehat{f'}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ such that $\sigma([\widehat{f}]) = [f]$, $\sigma([\widehat{f'}]) = [f']$, and $\widehat{WL}(\widehat{f}) = \widehat{WL}(\widehat{f'})$.

Proof of part (II) of Theorem 1.11. Let $[f], [f'] \in E^6(M_1 \sqcup M_2)$ be isotopy classes such that $WL(f) = WL(f')$. To complete the proof we need to find an embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ such that $[f'] \# [g] = [f]$.

Let $[\widehat{f}], [\widehat{f'}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ be the isotopy classes whose existence is guaranteed by Claim 2.8. By Haefliger Theorem 1.5 there is an embedding $g : S^3 \sqcup S^3 \rightarrow D_+^6$ such that

$$\sigma_R([\widehat{f'}]) \# [g] = \sigma_R([\widehat{f}]).$$

By $\#$ -commutativity Claim 2.2 we get

$$\sigma_R([\widehat{f'}] \# [g]) = \sigma_R([\widehat{f}]).$$

Clearly, $\widehat{WL}([\widehat{f'}] \# [g]) = \widehat{WL}([\widehat{f}])$, so by Bijectivity Lemma 2.7 we have

$$[\widehat{f'}] \# [g] = [\widehat{f}].$$

So,

$$\begin{aligned} [\widehat{f'}] \# [g] = [\widehat{f}] &\Rightarrow \sigma([\widehat{f'}] \# [g]) = \sigma([\widehat{f}]) \stackrel{(1)}{\Rightarrow} \\ &\stackrel{(1)}{\Rightarrow} \sigma([\widehat{f'}]) \# [g] = \sigma([\widehat{f}]) \stackrel{(2)}{\Rightarrow} [f'] \# [g] = [f], \end{aligned}$$

where (1) follows by the $\#$ -commutativity Claim 2.2 and (2) follows from the choice of $[\widehat{f}]$ and $[\widehat{f'}]$. We get that g is as required. \square

2.8. **Definition of ω .** Define

$$\omega_{k,i} : S^3 \rightarrow S^6 \quad \text{by the formula} \quad \omega_{k,i} := s_{k,i}|_{0 \times S^3},$$

see Fig.7.

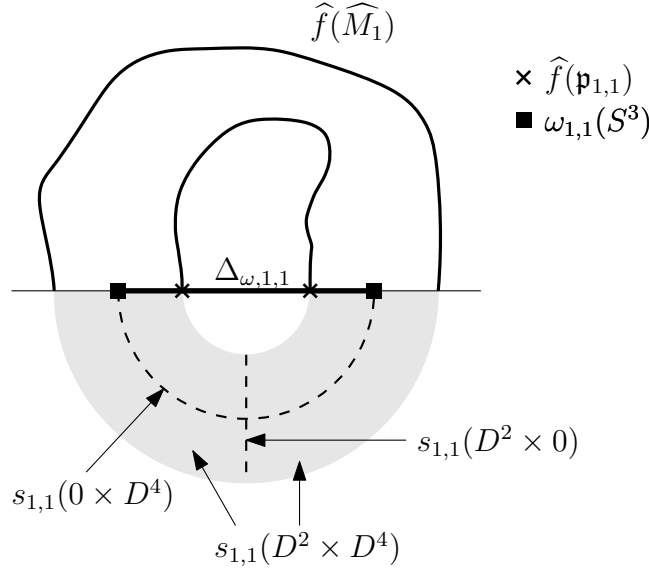


FIGURE 7. The circle $f(\mathbf{p}_{1,1})$, the sphere $\omega_{1,1}(S^3)$, and the disk $\Delta_{\omega,1,1}$.

2.9. **Multiple linked embedded connected sum.** Take any $[f] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ and $g : S^3 \rightarrow \partial D_+^6$ such that the images of \widehat{f} and g are disjoint. For $k \in \{1, 2\}$ we shall write

$$\widehat{f} \#_k g$$

meaning $\widehat{f} \#_k g'$ – the linked embedded connected sum of f with some embedding $g' : S^3 \rightarrow \text{Int} D_+^6$ obtained from g by a slight shift into the interior of D_+^6 . This agreement guarantees that $[\widehat{f} \#_k g] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$.

For any integer a we denote

- $\widehat{f} \#_k a g := \widehat{f} \#_k \underbrace{g \#_k g \dots \#_k g}_a$, if $a > 0$,
- $\widehat{f} \#_k a g := \widehat{f} \#_k (-a)(-g)$, if $a < 0$,
- $\widehat{f} \#_k a g := \widehat{f}$ if $a = 0$.

Here $-g : S^3 \rightarrow \partial D_+^6$ is an embedding such that $\text{Im}(-g) = \text{Im}(g)$ and $[g] \# [-g]$ is trivial considered as an isotopy class of an embedding $S^3 \rightarrow S^6$.

2.10. **Proof of part (III) of Theorem 1.11.** The following lemma allows us to describe the preimage of σ .

Lemma 2.9 (Preimage). *For any $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ we have that $\sigma([\widehat{f}]) = \sigma([\widehat{f}'])$ if and only if*

$$[\widehat{f}'] = [\widehat{f} \#_{i=1}^{m_1} a_i \omega_{1,i} \#_{i=1}^{m_1} b_i \omega_{i,1} \#_{j=1}^{m_2} c_j \omega_{2,j} \#_{j=1}^{m_2} d_j \omega_{2,j}]$$

for some integers a_i, b_i, c_j , and d_j .

Remark 2.10. In other words, the lemma states that $\sigma([\widehat{f}]) = \sigma([\widehat{f}'])$ if and only if $[\widehat{f}']$ can be obtained from $[\widehat{f}]$ by several operations of the form

- $[\widehat{f}] \rightarrow [\widehat{f} \#_1 \pm \omega_{1,i}]$,
- $[\widehat{f}] \rightarrow [\widehat{f} \#_2 \pm \omega_{1,i}]$,
- $[\widehat{f}] \rightarrow [\widehat{f} \#_1 \pm \omega_{2,j}]$,
- $[\widehat{f}] \rightarrow [\widehat{f} \#_2 \pm \omega_{2,j}]$.

where $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$.

Proof of the “if part” of Preimage Lemma 2.9. The remark above makes the “if” part obvious. For instance, there is an isotopy between $\sigma(\widehat{f} \#_1 \pm \omega_{1,i})$ and $\sigma(\widehat{f})$ which “drags” the sphere $\omega_{1,i}(S^3)$ along the disk $s_{1,i}(0 \times D^4)$. This is indeed an isotopy because the disk $s_{1,i}(0 \times D^4)$ is disjoint with $\text{Im}(\sigma(\widehat{f}))$, see Fig.7 for the case $i = 1$. \square

For a homology class $a \in H_1(\widehat{M}_k)$ we denote by $\text{lk}(\mathbf{p}_{k,i}, a)$ the linking number of $\mathbf{p}_{k,i} \subset \partial \widehat{M}_k$ and any oriented 1-submanifold of $\text{Int} \widehat{M}_k \subset S^3$ representing a . Clearly, this linking number is well defined, i.e., do not depend on the choice of the representative.

Denote by $[\mathbf{p}_{k,i}]$ the respective homology class in $H_1(\widehat{M}_k)$.

Let \widehat{f} be a proper embedding such that $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$. Denote

$$l_{k,i}(\widehat{f}) := \lambda(\omega_{k,i}, (\sigma_R \widehat{f})_k),$$

where $(\sigma_R \widehat{f})_k$ is the restriction of $\sigma_R \widehat{f} : S^3 \sqcup S^3 \rightarrow S^6$ to the k -th connected component of its domain.

Lemma 2.11 (Calculation). *Suppose that $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$, $1 \leq i \leq m_1$, and $1 \leq j \leq m_2$.*

In the first column of the table is an embedding \widehat{f}' . In the first row are symbols denoting different isotopy invariants.

In each cell of the columns “ λ_1 ” to “ r_2 ” is the difference of the corresponding invariant of $\sigma_R(\widehat{f}')$ and $\sigma_R(\widehat{f})$.

In each cell of the columns “ \widehat{W}_1 ” to “ \widehat{L}_2 ” is the difference of the corresponding invariant of \widehat{f}' and \widehat{f} .

\widehat{f}'	λ_1	λ_2	r_1	r_2	\widehat{W}_1	\widehat{W}_2	\widehat{L}_1	\widehat{L}_2
$\widehat{f} \#_1 \omega_{1,i}$	0	$2\text{lk}(\widehat{L}_1 \widehat{f}, \mathbf{p}_{1,i})$	$\frac{l_{1,i}(\widehat{f})}{2}$	0	$[\mathbf{p}_{1,i}]$	0	0	0
$\widehat{f} \#_2 \omega_{1,i}$	$2\text{lk}(\widehat{L}_1 \widehat{f}, \mathbf{p}_{1,i})$	$l_{1,i}(\widehat{f})$	0	0	0	0	$[\mathbf{p}_{1,i}]$	0
$\widehat{f} \#_2 \omega_{2,j}$	$2\text{lk}(\widehat{L}_2 \widehat{f}, \mathbf{p}_{2,j})$	0	0	$\frac{l_{2,j}(\widehat{f})}{2}$	0	$[\mathbf{p}_{2,j}]$	0	0
$\widehat{f} \#_1 \omega_{2,j}$	$l_{2,j}(\widehat{f})$	$2\text{lk}(\widehat{L}_2 \widehat{f}, \mathbf{p}_{2,j})$	0	0	0	0	0	$[\mathbf{p}_{2,j}]$

We shall refer to the cells of the table by their respective row number and column title. E.g., cell $(1, \lambda_1)$ contains 0 and means that $\lambda_1(\sigma_R(\widehat{f} \#_1 \omega_{1,i})) - \lambda_1(\sigma_R(\widehat{f})) = 0$; cell $(3, \widehat{W}_2)$ contains $[\mathfrak{p}_{2,j}]$ and means that $\widehat{W}_2(\widehat{f} \#_2 \omega_{2,j}) - \widehat{W}_2(\widehat{f}) = [\mathfrak{p}_{2,j}]$; etc.

Lemma 2.12 (Linking). *For any $k \in \{1, 2\}$, integers a_i , and isotopy class $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ the following implication holds*

$$\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}] = 0 \quad \Rightarrow \quad \sum_{i=1}^{m_k} a_i l_{k,i}(\widehat{f}) = \sum_{i=1}^{m_k} 2\text{lk}(\mathfrak{p}_{k,i}, \widehat{W}_k \widehat{f}).$$

Denote by $[\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}$ the respective homology class in $H_1(\partial \widehat{M}_k)$.

Consider the following part of the Mayer–Vietoris long exact sequence

$$H_2(M_k) \xrightarrow{\partial} H_1(\partial \widehat{M}_k) \xrightarrow{i_{\widehat{M}_k}, i_{T_{m_k}}} H_1(\widehat{M}_k) \oplus H_k(T_{m_1}),$$

where the maps $i_{\widehat{M}_k}$ and $i_{T_{m_k}}$ are induced by the inclusions $\partial \widehat{M}_k \subset \widehat{M}_k$ and $\partial \widehat{M}_k \subset T_{m_k}$.

Claim 2.13. *The image of $\partial : H_2(M_k) \rightarrow H_1(\partial \widehat{M}_k)$ is the subgroup of $H_1(\partial \widehat{M}_k)$ consisting of all linear combinations of the form $\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}$ such that $\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}] = 0 \in H_1(\widehat{M}_k)$.*

Proof. From the construction of M_k it is clear, that $\text{Ker}(i_{T_{m_k}})$ consists exclusively of all linear combinations of $[\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}$.

By the definition, $i_{\widehat{M}_k}([\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}) = [\mathfrak{p}_{k,i}]$, so any linear combination of the form $\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}$ is in $\text{Ker}(i_{\widehat{M}_k})$ if and only if $\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}] = 0 \in H_1(\widehat{M}_k)$.

Now the claim follows from the exactness of the Mayer–Vietoris sequence above. \square

Claim 2.14. *Take any $\alpha \in H_2(M_k)$. By Claim 2.13, $\partial \alpha = \sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}]_{\partial \widehat{M}_k}$ for some integers a_i . Then for any $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ and $[f] := \sigma([\widehat{f}])$*

$$\begin{aligned} \text{(I)} \quad L_k f \cap \alpha &= \sum_{i=1}^{m_k} a_i \text{lk}(\widehat{L}_k \widehat{f}, \mathfrak{p}_{k,i}), \\ \text{(II)} \quad W_k f \cap \alpha &= \sum_{i=1}^{m_k} a_i \frac{l_{k,i}}{2}. \end{aligned}$$

Proof. **(I).** Follows from

$$L_k f \cap \alpha = \widehat{L}_k \widehat{f} \cap \alpha = \text{lk}(\widehat{L}_k \widehat{f}, \partial \alpha) = \sum_{i=1}^{m_k} a_i \text{lk}(\widehat{L}_k \widehat{f}, \mathfrak{p}_{k,i}).$$

The first equality holds by Claim 2.5. The second equality holds by the definition of lk .

(II). Follows from

$$W_k f \cap \alpha = \widehat{W}_k \widehat{f} \cap \alpha = \text{lk}(\widehat{W}_k \widehat{f}, \partial \alpha) = \sum_{i=1}^{m_k} a_i \text{lk}(\widehat{W}_k \widehat{f}, \mathfrak{p}_{k,i}) = \sum_{i=1}^{m_k} a_i \frac{l_{k,i}}{2}.$$

The first equality holds by Claim 2.5. The second equality holds by the definition of lk . The last equality holds by Linking Lemma 2.12, which we can apply because $\sum_{i=1}^{m_k} a_i [\mathfrak{p}_{k,i}] = 0$ by Claim 2.13. \square

Proof of the “if” statement in part (III) of Theorem 1.11. Until the end of the proof identify $E^6(S^3 \sqcup S^3)$ with $\widetilde{\mathbb{Z}^4}$ by the isomorphism $\lambda_1 \times \lambda_2 \times r_1 \times r_2$.

Let $f : M_1 \sqcup M_2 \rightarrow S^6$ be an embedding. Let $g : S^3 \sqcup S^3 \rightarrow S^6$ be an embedding such that $[g] \in \text{Stab}_f \subset \widetilde{\mathbb{Z}^4}$. We need to prove that $[f] = [f] \# [g]$.

By the definition of Stab_f , there are $\alpha, \beta \in H_2(M_1)$ and $\gamma, \delta \in H_2(M_2)$ such that $[g] = [g_\alpha] + [g_\beta] + [g_\gamma] + [g_\delta]$, where

- $[g_\alpha] = (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) \in \widetilde{\mathbb{Z}^4}$,
- $[g_\beta] = (2L_1f \cap \beta, 2W_1f \cap \beta, 0, 0) \in \widetilde{\mathbb{Z}^4}$,
- $[g_\gamma] = (2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) \in \widetilde{\mathbb{Z}^4}$,
- $[g_\delta] = (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \in \widetilde{\mathbb{Z}^4}$.

It is enough to prove that $[f] = [f] \# [g_\alpha]$, $[f] = [f] \# [g_\beta]$, $[f] = [f] \# [g_\gamma]$, and $[f] = [f] \# [g_\delta]$. We shall only prove the first equality because the proofs of others are analogous.

By Claim 2.13, there are integers a_i such that $\partial\alpha = \sum_{i=1}^{m_1} a_i [\mathbf{p}_{1,i}]_{\partial\widehat{M}_1}$. By Surjection Lemma 2.3, there is an embedding $\widehat{f} : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$ such that $\sigma([\widehat{f}]) = [f]$. Denote

$$[\widehat{f}'] := [\widehat{f} \#_{i=1}^{m_1} a_i \omega_{1,i}].$$

Now the equality $[f] = [f] \# [g_\alpha]$, which we want to prove, follows from

$$[f] = \sigma([\widehat{f}]) \stackrel{(1)}{=} \sigma([\widehat{f}']) \stackrel{(2)}{=} \sigma([\widehat{f}] \# [\widehat{g}_\alpha]) \stackrel{(3)}{=} \sigma([\widehat{f}]) \# [g_\alpha] = [f] \# [g_\alpha],$$

where (1) follows by Preimage Lemma 2.9 and (3) follows by $\#$ -commutativity Claim 2.2. Equation (2) follows from

$$\widehat{WL}([\widehat{f}']) \stackrel{(4)}{=} \widehat{WL}([\widehat{f}] \# [\widehat{g}_\alpha])$$

and

$$\sigma_R([\widehat{f}']) \stackrel{(5)}{=} \sigma_R([\widehat{f}] \# [\widehat{g}_\alpha])$$

by Bijection Lemma 2.7. It remains to prove (4) and (5).

Now (4) follows from

$$\widehat{WL}([\widehat{f}']) - \widehat{WL}([\widehat{f}] \# [\widehat{g}_\alpha]) = \widehat{WL}([\widehat{f}']) - \widehat{WL}([\widehat{f}]) = \sum_{i=1}^{m_1} a_i [\mathbf{p}_{1,i}] = 0,$$

where the second equality follows by the definition of $[\widehat{f}']$ and Calculation Lemma 2.11, cells $(1, \widehat{W}_1 - \widehat{L}_2)$. The last equality holds by Claim 2.13.

And (5) follows from

$$\begin{aligned} \sigma_R([\widehat{f}']) - \sigma_R([\widehat{f}] \# [\widehat{g}_\alpha]) &= \sigma_R([\widehat{f}']) - \sigma_R([\widehat{f}]) - [g_\alpha] = \\ &= (0, 2 \sum_{i=1}^{m_1} a_i \text{lk}(\widehat{L}_1 \widehat{f}, \mathbf{p}_{1,i}), \sum_{i=1}^{m_1} a_i \frac{l_{1,i}(\widehat{f})}{2}, 0) - [g_\alpha] = \\ &= (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) - (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) = 0 \in \widetilde{\mathbb{Z}^4}, \end{aligned}$$

where the second equality holds by Calculation Lemma 2.11, cells $(1, \lambda_1 - r_2)$. The third equality holds by Claim 2.14 and by the definition of $[g_\alpha]$. \square

Proof of the “only if” statement in part (III) of Theorem 1.11. Until the end of the proof identify $E^6(S^3 \sqcup S^3)$ with $\widetilde{\mathbb{Z}^4}$ by the isomorphism $\lambda_1 \times \lambda_2 \times r_1 \times r_2$.

Let $f : M_1 \sqcup M_2 \rightarrow S^6$ be an embedding. Let $g : S^3 \sqcup S^3 \rightarrow S^6$ be an embedding such that $[f] = [f] \# [g]$. We need to prove that $[g] \in \text{Stab}_f$.

By Surjection Lemma 2.3, there is an embedding $\widehat{f} : \widehat{M}_1 \sqcup \widehat{M}_2 \rightarrow D_+^6$ such that $\sigma([\widehat{f}]) = [f]$. By $\#$ -commutativity Claim 2.2, $\sigma([\widehat{f}] \# [g]) = \sigma([\widehat{f}]) \# [g] = [f] \# [g] = [f]$.

So both $[\widehat{f}]$ and $[\widehat{f}] \# [g]$ are σ -preimages of $[f]$. By Preimage Lemma 2.9, there are integers a_i, b_i, c_j , and d_j such that

$$[\widehat{f}] \# [g] = [\widehat{f} \#_1^{m_1} a_i \omega_{1,i} \#_2^{m_1} b_i \omega_{i,1} \#_2^{m_2} c_j \omega_{2,j} \#_1^{m_2} d_j \omega_{2,j}].$$

Clearly $\widehat{WL}([\widehat{f}]) = \widehat{WL}([\widehat{f}] \# [g])$. From that, the equation above, and Calculation Lemma 2.11 (last four columns of the table) we get that

$$\sum_{i=1}^{m_1} a_i [\mathbf{p}_{1,i}] = \sum_{i=1}^{m_1} b_i [\mathbf{p}_{1,i}] = \sum_{j=1}^{m_2} c_j [\mathbf{p}_{2,j}] = \sum_{j=1}^{m_2} d_j [\mathbf{p}_{2,j}] = 0.$$

By Claim 2.13, there is $\alpha \in H_2(M_1)$ such that $\partial\alpha = \sum_{i=1}^{m_1} a_i [\mathbf{p}_{1,i}]_{\partial\widehat{M}_1}$. The definitions of $\beta \in H_2(M_1)$, $\gamma, \delta \in H_2(M_2)$ are analogous but with $(a_i, \mathbf{p}_{1,i})$ replaced by $(b_i, \mathbf{p}_{1,i})$, $(c_j, \mathbf{p}_{2,j})$, and $(d_j, \mathbf{p}_{2,j})$, respectively.

The statement of the theorem now follows from

$$\begin{aligned} \sigma_R([\widehat{f}]) + [g] &\stackrel{(1)}{=} \sigma_R([\widehat{f}] \# [g]) = \\ &= \sigma_R([\widehat{f} \#_1^{m_1} a_i \omega_{1,i} \#_2^{m_1} b_i \omega_{i,1} \#_2^{m_2} c_j \omega_{2,j} \#_1^{m_2} d_j \omega_{2,j}]) \stackrel{(2)}{=} \\ &\stackrel{(2)}{=} \sigma_R([\widehat{f} \#_1^{m_1} a_i \omega_{1,i} \#_2^{m_1} b_i \omega_{i,1} \#_2^{m_2} c_j \omega_{2,j}]) + (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \stackrel{(3)}{=} \\ &\stackrel{(3)}{=} \sigma_R([\widehat{f} \#_1^{m_1} a_i \omega_{1,i} \#_2^{m_1} b_i \omega_{i,1}]) + (2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) + (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \stackrel{(4)}{=} \\ &\stackrel{(4)}{=} \sigma_R([\widehat{f} \#_1^{m_1} a_i \omega_{1,i}]) + (2L_1f \cap \beta, 2W_1f \cap \beta, 0, 0) + \\ &\quad (2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) + (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \stackrel{(5)}{=} \\ &\stackrel{(5)}{=} \sigma_R([\widehat{f}]) + (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) + (2L_1f \cap \beta, 2W_1f \cap \beta, 0, 0) + \\ &\quad + (2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) + (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \\ &\quad \downarrow \\ [g] &= (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) + (2L_1f \cap \beta, 2W_1f \cap \beta, 0, 0) + \\ &\quad + (2L_2f \cap \gamma, 0, 0, W_2f \cap \gamma) + (2W_2f \cap \delta, 2L_2f \cap \delta, 0, 0) \in \text{Stab}_f. \end{aligned}$$

Here (1) follows by $\#$ -commutativity Claim 2.2. It remains to prove (2-5). Let us only prove (5) as (2-4) are proved analogously. Equation (5) is equivalent to

$$\sigma_R([\widehat{f} \#_1^{m_1} a_i \omega_{1,i}]) - \sigma_R([\widehat{f}]) = (0, 2L_1f \cap \alpha, W_1f \cap \alpha, 0) \in \widetilde{\mathbb{Z}^4},$$

which in turn follows by Calculation Lemma 2.11, cells $(1, \lambda_1 - r_2)$, and by Claim 2.13. \square

3. PROOF OF SURJECTIVITY, BIJECTIVITY, AND PREIMAGE LEMMAS 2.3, 2.7, 2.9.

Throughout this section we denote by $q_{k,i}$ the circle $S^1 \times 0$ in the i -th connected component of T_{m_k} , see Fig.8.

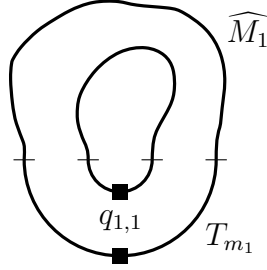


FIGURE 8. The circle $q_{1,1}$ represented by a pair of squares.

3.1. Proof of Surjectivity Lemma 2.3.

Proof of Surjectivity Lemma 2.3. We only prove that the map σ is surjective. The map σ_R is surjective by an analogous argument.

Choose an arbitrary embedding $f : M_1 \sqcup M_2 \rightarrow S^6$. By general position, there are 2-disks $\Delta_{k,i}$ in S^6 (see Fig.9), such that

- $\partial\Delta_{k,i} = f(q_{k,i})$,
- interiors of all $\Delta_{k,i}$ are pairwise disjoint and are disjoint with $f(M_1 \sqcup M_2)$.

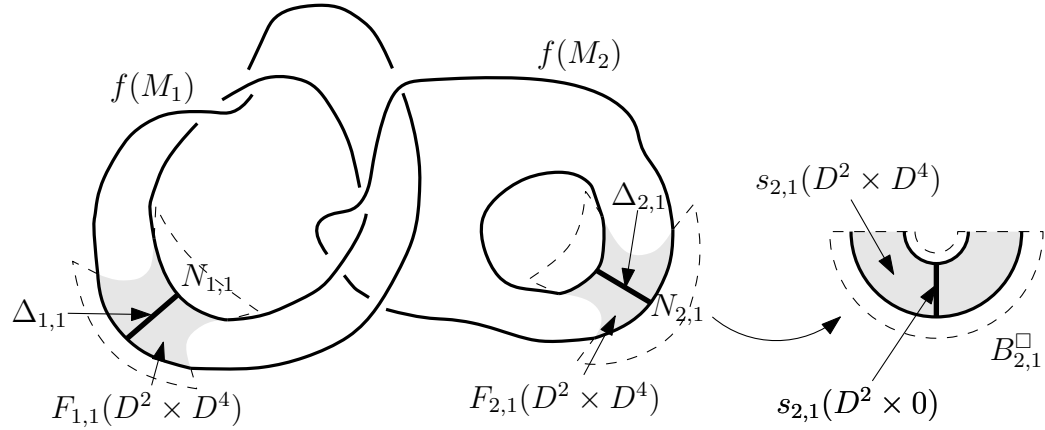


FIGURE 9. Proof of Surjectivity Lemma 2.3.

The restriction of f to the i -th component of T_{m_k} can be extended to an embedding $F_{k,i} : D^2 \times D^4 \rightarrow S^6$ such that (see Fig.9)

- $F_{k,i}(D^2 \times 0) = \Delta_{k,i}$,
- $\text{Im}(F_{k,i}) \cap \text{Im}(f) = F_{k,i}(S^1 \times D^2)$,

- images of all the $F_{k,i}$ are pairwise disjoint.

Indeed, the obstruction to an extension to $D^2 \times D^2$ is in $\pi_1(V_{4,2}) = 0$. Having $F_{k,i}$ already defined on $D^2 \times D^2$ we can extend it to $D^2 \times D^4$ without any obstructions.

Let $N_{k,i}$ be a tubular neighbourhood modulo $F_{k,i}(D^2 \times S^3)$ of $F_{k,i}(D^2 \times D^4)$ (see Fig.9). We can choose all $N_{k,i}$ to be pairwise disjoint and such that $N_{k,i} \cap f(\widehat{M}_1 \sqcup \widehat{M}_2) = F_{k,i}(S^1 \times D^2)$. By construction, $F_{k,i} : D^2 \times D^4 \rightarrow N_{k,i}$ is isotopic to the composition of $s_{k,i} : D^2 \times D^4 \rightarrow B_{k,i}^\square$ with some diffeomorphism $B_{k,i}^\square \rightarrow N_{k,i}$, see the right part of Fig.9. There is a 6-ball B containing all of $N_{k,i}$, interior of B being also disjoint with $f(\widehat{M}_1 \sqcup \widehat{M}_2)$.

Apply an ambient isotopy of S^6 which maps B to D_-^6 , each $N_{k,i}$ to $B_{k,i}^\square$, and each $F_{k,i}$ to $s_{k,i}$.

Denote by f' the result obtained from f by the isotopy. By construction, f' is in the image of σ . \square

3.2. Proof of the “only if” part of Preimage Lemma 2.9. We need the following Claim to prove the “only if” part of Preimage Lemma 2.9.

Claim 3.1. *Let $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ be isotopy classes. Suppose that embeddings $\sigma(\widehat{f})$ and $\sigma(\widehat{f}')$ are isotopic. Then there is a concordance between $\sigma(\widehat{f})$ and $\sigma(\widehat{f}')$ fixed on $T_{m_1} \sqcup T_{m_2}$.*

Proof. Denote $f := \sigma(\widehat{f})$ and $f' := \sigma(\widehat{f}')$. By the definition of σ , we have that $f|_{T_{m_1} \sqcup T_{m_2}} = f'|_{T_{m_1} \sqcup T_{m_2}} = s_1 \sqcup s_2$.

Clearly, it suffices to find a concordance between f and f' fixed on some tubular neighbourhood of each circle $q_{k,i}$.

Let $F : (M_1 \sqcup M_2) \times I \rightarrow S^6$ be an isotopy between f and f' . By general position, F is isotopic relative to the boundary to some concordance F' fixed on each $q_{k,i}$.

At each point of $F'(q_{1,1} \times I)$ identify with \mathbb{R}^5 the normal to $F'(q_{1,1} \times I)$ space in $S^6 \times I$. The restriction of F' to a small tubular neighbourhood of $q_{1,1} \times I$ gives us then a map $u : S^1 \times I \rightarrow V_{5,2}$. We can choose the identification so that u is constant on $S^1 \times \partial I$.

Let $\bar{u} : \frac{S^1 \times I}{S^1 \times \partial I} \rightarrow V_{5,2}$ be the quotient map. Space $\frac{S^1 \times I}{S^1 \times \partial I}$ is homotopically equivalent to $S^2 \vee S^1$. From $\pi_2(V_{5,2}) = \pi_1(V_{5,2}) = 0$ it follows that \bar{u} is null-homotopic. Therefore u is homotopic relative $S^1 \times \partial I$ to the constant map.

It implies that isotopying F' in a small tubular neighbourhood of $q_{1,1} \times I$ we can make F' constant on this tubular neighbourhood. Doing this for all k, i we get the required concordance. \square

Proof of the “only if” part of Preimage Lemma 2.9. Suppose that $\sigma([\widehat{f}]) = \sigma([\widehat{f}'])$ for some $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$. Denote $f := \sigma(\widehat{f})$ and $f' := \sigma(\widehat{f}')$.

By Claim 3.1, there is a concordance F between f and f' , fixed on $T_{m_1} \sqcup T_{m_2}$.

Denote $\Delta_{k,i} := s_{k,i}(D^2 \times 0)$. Disks $\Delta_{k,i}$ are pairwise disjoint, $\partial \Delta_{k,i} = f(q_{k,i}) = f'(q_{k,i}) = F_t(q_{k,i})$ for every $t \in I$, the interior of each $\Delta_{k,i}$ is disjoint with $s_1(T_{m_1}) \sqcup s_2(T_{m_2})$.

For any n and any two general position submanifolds $A, B \subset S^n$, $\dim A + \dim B = n$, denote by $\#|A \cap B|$ the algebraic number of points of intersection $A \cap B$. For each $\Delta_{k,i}$ denote by $\Delta_{k,i}$ its interior.

For $1 \leq i \leq m_1$, $1 \leq j \leq m_2$ define

$$\begin{aligned} a_i &:= \#|\Delta_{1,i}^\circ \times I \cap F(M_1 \times I)|, & b_i &:= \#|\Delta_{1,i}^\circ \times I \cap F(M_2 \times I)|, \\ c_j &:= \#|\Delta_{2,j}^\circ \times I \cap F(M_2 \times I)|, & d_j &:= \#|\Delta_{2,j}^\circ \times I \cap F(M_1 \times I)|. \end{aligned}$$

Denote

$$\widehat{f''} := \widehat{f} \#_{i=1}^{m_1} a_i \omega_{1,i} \#_{i=1}^{m_1} b_i \omega_{i,1} \#_{j=1}^{m_2} c_j \omega_{2,j} \#_{j=1}^{m_2} d_j \omega_{2,j},$$

and

$$f'' := \sigma(\widehat{f''}).$$

It remains to prove that $[\widehat{f'}] = [\widehat{f''}]$.

By construction, there is an isotopy F'' between f and f'' which “drags” spheres $\omega_{1,i}$ and $\omega_{2,j}$ along pairwise disjoint embedded disks $s_{1,i}(0 \times D^4)$ and $s_{2,j}(0 \times D^4)$ for all i and j . Isotopy F'' is fixed on $T_{m_1} \sqcup T_{m_2}$. We have that

$$\begin{aligned} \#|\Delta_{1,i}^\circ \times I \cap F''(M_1 \times I)| &= a_i, & \#|\Delta_{1,i}^\circ \times I \cap F''(M_2 \times I)| &= b_i, \\ \#|\Delta_{2,j}^\circ \times I \cap F''(M_2 \times I)| &= c_j, & \#|\Delta_{2,j}^\circ \times I \cap F''(M_1 \times I)| &= d_j. \end{aligned}$$

Consider now the concordance $H := -F \cup F''$ between f' and f'' . By construction, H is fixed on $T_{m_1} \sqcup T_{m_2}$ and

$$\begin{aligned} \#|\Delta_{1,i}^\circ \times I \cap H(M_1 \times I)| &= 0, & \#|\Delta_{1,i}^\circ \times I \cap H(M_2 \times I)| &= 0, \\ \#|\Delta_{2,j}^\circ \times I \cap H(M_2 \times I)| &= 0, & \#|\Delta_{2,j}^\circ \times I \cap H(M_1 \times I)| &= 0. \end{aligned}$$

So, using the Whitney trick ([Mi65, Theorem 6.6]), we can isotope H , changing it only on $(\widehat{M}_1 \sqcup \widehat{M}_2) \times \text{Int} I$, to some concordance H' whose image is disjoint with each $\Delta_{1,i}^\circ \times I$ and $\Delta_{2,j}^\circ \times I$.

Now we can “push” the image of H' away from each $\Delta_{1,i} \times I$ along the vectors of the normal framing of $\Delta_{1,i} \times I$ given by the embeddings $s_{1,i}(D^2 \times D^4)$ (recall that $s_{k,i}(D^2 \times 0) = \Delta_{k,i}$ by the definition of $\Delta_{k,i}$). Likewise we can “push” the image of H' away from each $\Delta_{2,j} \times I$.

We obtain a new concordance H'' between f' and f'' such that $H''((\widehat{M}_1 \sqcup \widehat{M}_2) \times I) \subset D_+^6 \times I$. The restriction of H'' to $(\widehat{M}_1 \sqcup \widehat{M}_2) \times I$ is a concordance between $\widehat{f'}$ and $\widehat{f''}$ in D_+^6 fixed on the boundary. In codimension at least 3 concordance implies isotopy, see [Hu70, Theorem 2.1], therefore $[\widehat{f'}] = [\widehat{f''}]$. \square

3.3. Proof of Bijectivity Lemma 2.7. To prove the Bijectivity Lemma 2.7 we shall need the following analogue of Preimage Lemma 2.9.

For all $k \in \{1, 2\}$ and $1 \leq i \leq m_k$ define

$$\omega_{R,k,i} : S^3 \rightarrow S^6 \quad \text{by the formula} \quad \omega_{R,k,i} := s_{R,k,i}|_{0 \times S^3}.$$

Lemma 3.2 (Preimage'). *For any $[\widehat{f}], [\widehat{f'}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ we have that $\sigma_R([\widehat{f}]) = \sigma_R([\widehat{f'}])$ if and only if*

$$[\widehat{f'}] = [\widehat{f} \#_{i=1}^{m_1} a_i \omega_{R,1,i} \#_{i=1}^{m_1} b_i \omega_{R,i,1} \#_{j=1}^{m_2} c_j \omega_{R,2,j} \#_{j=1}^{m_2} d_j \omega_{R,2,j}]$$

for some integers a_i , b_i , c_j , and d_j .

The proof of Preimage' Lemma 3.2 is analogous to the proof of Preimage Lemma 2.9.

Proof of Bijectivity Lemma 2.7. Let us first prove that the restriction $\sigma_R|_{\widehat{WL}^{-1}(x)}$ is surjective.

Choose any $[g] \in E^6(S^3 \sqcup S^3)$. The map \widehat{WL} is surjective, which is proved analogously to the surjectivity of WL (part (I) of Theorem 1.11). So, we can choose some $[\widehat{f}] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ such that $\widehat{WL}(\widehat{f}) = x$.

There is an isotopy class $[g'] \in E^6(S^3 \sqcup S^3)$ such that $\sigma_R([\widehat{f}]) \# [g'] = [g]$. Then $\sigma_R([\widehat{f}] \# [g']) = [g]$ and $\widehat{WL}([\widehat{f}] \# [g']) = \widehat{WL}([\widehat{f}]) = x$.

Let us now prove that the restriction $\sigma_R|_{\widehat{WL}^{-1}(x)}$ is injective. Let $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ be some isotopy classes such that $\sigma_R([\widehat{f}]) = \sigma_R([\widehat{f}'])$ and $\widehat{WL}(\widehat{f}) = \widehat{WL}(\widehat{f}') = x$.

By Preimage' Lemma 3.2, we have that

$$[\widehat{f}'] = [\widehat{f} \#_{i=1}^{m_1} a_i \omega_{R,1,i} \#_{i=1}^{m_1} b_i \omega_{R,i,1} \#_{j=1}^{m_2} c_j \omega_{R,2,j} \#_{j=1}^{m_2} d_j \omega_{R,2,j}]$$

for some integers a_i, b_i, c_j , and d_j .

Similarly to \mathbf{m} and $\mathbf{p}_{k,i}$ denote

$$\mathbf{p} := S^1 \times * \subset S^1 \times D^2$$

and

$$\mathbf{m}_{k,i} := P_{k,i} R \mathbf{p}.$$

By $[\mathbf{m}_{1,i}]$ we denote the corresponding homology classes in \widehat{M}_1 (analogously to $[\mathbf{p}_{1,i}]$). Let us compute $\text{lk}(\sum_{i=1}^{m_1} a_i [\mathbf{m}_{1,i}], P_{1,1} q_{1,1})$ in two ways.

Circles $\mathbf{m}_{k,i}$ are meridians of the pairwise disjoint embedded solid tori $P_{k,i}(S^1 \times D^2) \subset S^3$ and $P_{1,1} q_{1,1}$ is the parallel of the solid torus $P_{1,1}(S^1 \times D^2) \subset S^3$. Therefore,

$$\text{lk}(\sum_{i=1}^{m_1} a_i [\mathbf{m}_{1,i}], P_{1,1} q_{1,1}) = a_1.$$

By the analogue of Calculation Lemma 2.11 (cell $(1, W_1)$), we have that

$$\widehat{W}_1(\widehat{f}') = \widehat{W}_1(\widehat{f}) + \sum_{i=1}^{m_1} a_i [\mathbf{m}_{1,i}].$$

Since $\widehat{W}_1(\widehat{f}') = \widehat{W}_1(\widehat{f})$, it follows that $\sum_{i=1}^{m_1} a_i [\mathbf{m}_{1,i}] = 0 \in H_1(\widehat{M}_1)$. Circle $P_{1,1} q_{1,1} \subset S^3$ is disjoint with $\widehat{M}_1 \subset S^3$, therefore

$$\text{lk}(\sum_{i=1}^{m_1} a_i [\mathbf{m}_{1,i}], P_{1,1} q_{1,1}) = 0.$$

Combining the last two paragraphs we get that $a_1 = 0$. By the same argument, $a_i = b_j = c_j = d_j = 0$ for all i, j , meaning that $[\widehat{f}] = [\widehat{f}']$. \square

4. PROOF OF CALCULATION LEMMA 2.11.

For the proof of Calculation Lemma 2.11 we use Lemma 4.1 which can be seen as an alternative definition of the linking coefficients λ_1 and λ_2 . We shall also need additional Claim 4.2.

4.1. Definition of framed intersections and preimages. Let $A, B \subset S^n$ be submanifolds in general position. Suppose that B is framed. Then the *framed intersection* $A \cap B$ is a framed submanifold of A . The framing of $A \cap B \subset A$ is obtained by the projection of the framing of B onto the tangent space of A and subsequent Gram-Schmidt orthonormalising process.

Let $f : A \rightarrow S^n$ be an embedding and let $a \subset f(A)$ be a framed submanifold of $f(A)$. Then $f^{-1}(a)$ is called a *framed preimage* of a . The framing of $f^{-1}(a)$ is the df^{-1} -image of the framing of a .

Recall that h denotes the Hopf invariant of a framed 1-submanifold of S^3 .

Lemma 4.1. *Let $g : S_1^3 \sqcup S_2^3 \rightarrow S^6$ be an embedding, where S_1^3 and S_2^3 are two distinct copies of S^3 . Suppose that the restriction of g to each connected component is trivial. Let Δ_1, Δ_2 be framed embedded disks in general position bounded by gS_1^3 and gS_2^3 , respectively. Then*

$$\lambda_1(g) = h(g^{-1}(gS_1^3 \cap \Delta_2)) \quad \text{and} \quad \lambda_2(g) = h(g^{-1}(gS_2^3 \cap \Delta_1)).$$

Proof. We only prove the first claim as the second one is analogous. Clearly, Δ_2 is the preimage of a regular point of some homotopy equivalence $S^6 \setminus gS_2^3 \rightarrow S^2$. Therefore $gS_1^3 \cap \Delta_2$ is the preimage of the same point under the restriction of this homotopy equivalence to gS_1^3 . The first claim of the lemma now holds by the definition of λ_1 . \square

4.2. Definition of $\Delta_{\omega,k,i}$. Let $\Delta_{\omega,k,i} \subset \partial D_-^6$ be an embedded framed 4-disk bounded by $\omega_{k,i}(S^3)$ and such that for any $[\hat{f}] \in \hat{E}^6(\hat{M}_1 \sqcup \hat{M}_2)$

$$\hat{f}^{-1}(\hat{f}\hat{M}_k \cap \Delta_{\omega,k,i}) = (\sigma_R \hat{f})^{-1}(\sigma_R \hat{f}(S_k^3) \cap \Delta_{\omega,k,i}) = \mathfrak{p}_{k,i},$$

where S_k^3 is the k -th component of the domain of $\sigma_R f$, see Fig.7. Here the “=” signs mean the equality of both sides as framed submanifolds. The first equality holds by definition of σ_R and the second equality is a part of the definition of $\Delta_{\omega,k,i}$.

Claim 4.2. *For any $k \in \{1, 2\}$, $1 \leq i \leq m_k$ there exist a disk $\Delta_{\omega,k,i} \subset \partial D_-^6$ satisfying the properties above.*

The claim is made obvious by Fig.7.

4.3. Proof of Calculation Lemma 2.11. We shall prove the first two rows of the table, the proof for the second two rows is analogous. Without a loss of generality we may assume that $i = 1$. For brevity denote $\omega := \omega_{1,1}$ and $\Delta_\omega := \Delta_{\omega,1,1}$.

Without a loss of generality we may also assume that the restriction of $\sigma_R \hat{f}$ to the second component is trivial. Indeed, for any $g : S^3 \rightarrow D_+^6$ whose image is far away from the images of \hat{f} and \hat{f}' we may substitute \hat{f} and \hat{f}' by $\hat{f} \#_2 g$ and $\hat{f}' \#_2 g$, respectively, without changing any of the table entries. By choosing g appropriately we can always make the restriction of $\sigma_R \hat{f}$ to the second component trivial.

Let $F_k : S^3 \rightarrow S^6$ be the restriction of $\sigma_R f$ to the k -th component. Embedding F_2 is trivial by the argument in the previous paragraph.

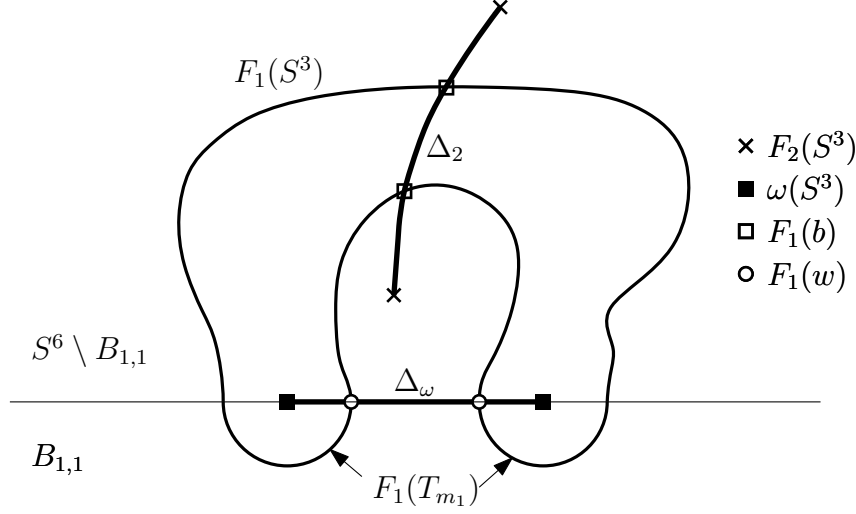


FIGURE 10. Proof of Calculation Lemma 2.11.

Consider some embedded framed 4-disk Δ_2 in the complement to $B_{1,1}$ bounded by $F_2(S^3)$. Denote

$$w := F_1^{-1}(F_1(S^3) \cap \Delta_\omega),$$

$$b := F_1^{-1}(F_1(S^3) \cap \Delta_2).$$

Both w and b are framed 1-submanifolds of S^3 . Recall that $w = \mathbf{p}_{1,1}$ as a framed submanifold by Claim 4.2.

We prove the second row of the table first.

Cell (2, λ_1). In this cell we need to compute $\lambda_1(\sigma_R(\widehat{f} \#_2 \omega_{1,1})) - \lambda_1(\sigma_R(\widehat{f})) = \lambda(F_1, F_2 \# \omega) - \lambda(F_1, F_2)$.

The disks Δ_2 and Δ_ω are disjoint by construction. So there is a framed embedded disk $\Delta_{F_2 \# \omega}$, bounded by $(F_2 \# \omega)(S^3)$ and such that $F_1 S^3 \cap \Delta_{F_2 \# \omega} = (F_1 S^3 \cap \Delta_2) \sqcup (F_1 S^3 \cap \Delta_\omega)$. So by Lemma 4.1 we have

$$\begin{aligned} \lambda(F_1, F_2 \# \omega) - \lambda(F_1, F_2) &= h(F_1^{-1}(F_1 \cap \Delta_{F_2 \# \omega})) - h(F_1^{-1}(F_1 \cap \Delta_2)) = \\ &= h(b \sqcup w) - h(b) = h(b) + h(w) + 2\text{lk}(b, w) - h(b) = h(w) + 2\text{lk}(b, w) = \\ &= h(\mathbf{p}_{1,1}) + 2\text{lk}(b, \mathbf{p}_{1,1}) = 0 + 2\text{lk}(b, \mathbf{p}_{1,1}) = 2\text{lk}(\widehat{L}_1 \widehat{f}, \mathbf{p}_{1,1}). \end{aligned}$$

The equation before the last holds because $h(\mathbf{p}_{1,1}) = 0$ by Claim 2.1. The last equation holds by Claim 2.4 (take $\Delta_2 \cap D_+^6$ as “ Δ ” in the statement of the claim. Clearly, $\Delta_2 \cap D_+^6$ satisfies the necessary condition by construction.).

Cell (2, λ_2). In this cell we need to compute $\lambda_2(\sigma_R(\widehat{f} \#_2 \omega_{1,1})) - \lambda_2(\sigma_R(\widehat{f})) = \lambda(F_2 \# \omega, F_1) - \lambda(F_2, F_1)$.

We have

$$\lambda(F_2 \# \omega, F_1) - \lambda(F_2, F_1) = \lambda(F_2, F_1) + \lambda(\omega, F_1) - \lambda(F_2, F_1) = \lambda(\omega, F_1) = l_{1,1}(\widehat{f}).$$

The first equation holds by Lemma 1.20. The last equation is the definition of $l_{1,1}(\widehat{f})$.

Cell (2, r_1). In this cell we have $r_1(\sigma_R(\widehat{f}\#_2\omega_{1,1})) - r_1(\sigma_R(\widehat{f})) = 0$ because the restrictions of $\sigma_R(\widehat{f}\#_2\omega_{1,1})$ and $\sigma_R(\widehat{f})$ to the first connected component are the same by the definition of $\#_2$.

Cell (2, r_2). By construction ω is trivial and the images of ω and F_2 lie in disjoint 6-balls. So in this cell we have $r_2(\sigma_R(\widehat{f}\#_2\omega_{1,1})) - r_2(\sigma_R(\widehat{f})) = r(F_2\#\omega) - r(F_2) = 0$.

Cells (2, $\widehat{W}_1\text{-}\widehat{L}_2$). Clearly, there is a homotopy between $\widehat{f}\#_2\omega$ and \widehat{f} which shrinks $\omega(S^3)$ along the disk Δ_ω . The disk Δ_ω is disjoint with the image of \widehat{M}_2 and the homotopy is the identity on \widehat{M}_1 . So $\widehat{f}' = \widehat{f}\#_2\omega$ and \widehat{f} differ at only one Whitney invariant out of four, namely

$$\widehat{L}_1(\widehat{f}\#_2\omega) - \widehat{L}_1(\widehat{f}) = \widehat{f}^{-1}[\widehat{f}(\widehat{M}_1) \cap \Delta_\omega] = [w] = [\mathbf{p}_{1,1}].$$

Cell (1, λ_1). In this cell we need to compute $\lambda_1(\sigma_R(\widehat{f}\#_1\omega_{1,1})) - \lambda_1(\sigma_R(\widehat{f})) = \lambda(F_1\#\omega, F_2) - \lambda(F_1, F_2)$.

We have

$$\lambda(F_1\#\omega, F_2) - \lambda(F_1, F_2) = \lambda(F_1, F_2) + \lambda(\omega, F_2) - \lambda(F_1, F_2) = \lambda(\omega, F_2) = 0.$$

The first equation holds by Lemma 1.20. The last equation holds because the images of ω and F_2 lie in disjoint 6-balls.

Cell (1, λ_2). In this cell we need to compute $\lambda_2(\sigma_R(\widehat{f}\#_1\omega_{1,1})) - \lambda_2(\sigma_R(\widehat{f})) = \lambda(F_2, F_1\#\omega) - \lambda(F_2, F_1)$.

By Lemma 1.22 we have

$$2r(F_1\#F_2\#\omega) = \lambda(F_2, F_1\#\omega) +$$

$$2r(F_1\#F_2\#\omega) = \lambda(F_2\#\omega, F_1) + \lambda(F_1, F_2\#\omega) + 2r(F_2\#\omega) + 2r(F_1).$$

So

$$\lambda(F_2, F_1\#\omega) = \lambda(F_2\#\omega, F_1) + \lambda(F_1, F_2\#\omega) + 2r(F_2\#\omega) + 2r(F_1) - \lambda(F_1\#\omega, F_2) - 2r(F_2) - 2r(F_1\#\omega).$$

Applying Lemma 1.20 and Lemma 1.22 we get

$$\begin{aligned} \lambda(F_2, F_1\#\omega) &= \lambda(F_2\#\omega, F_1) + \lambda(F_1, F_2\#\omega) + 2r(F_2\#\omega) + 2r(F_1) - \lambda(F_1\#\omega, F_2) - 2r(F_2) - 2r(F_1\#\omega) = \\ &= \lambda(F_2, F_1) + \lambda(\omega, F_1) + \lambda(F_1, F_2\#\omega) + 2r(F_2) + 2r(\omega) + \lambda(F_2, \omega) + \lambda(\omega, F_2) + 2r(F_1) - \\ &\quad - \lambda(F_1, F_2) - \lambda(\omega, F_2) - 2r(F_2) - 2r(F_1) - 2r(\omega) - \lambda(F_1, \omega) - \lambda(\omega, F_1) = \\ &= \lambda(F_2, F_1) + \lambda(F_1, F_2\#\omega) + \lambda(F_2, \omega) - \lambda(F_1, F_2) - \lambda(F_1, \omega) = \\ &= \lambda(F_2, F_1) + \lambda(F_1, F_2\#\omega) - \lambda(F_1, F_2) - \lambda(F_1, \omega), \end{aligned}$$

where the last equation holds because $\lambda(F_2, \omega) = 0$ (see paragraph “Cell (1, λ_1)”). So

$$\begin{aligned} \lambda(F_2, F_1\#\omega) - \lambda(F_2, F_1) &= \lambda(F_2, F_1) + \lambda(F_1, F_2\#\omega) - \lambda(F_1, F_2) - \lambda(F_1, \omega) - \lambda(F_2, F_1) = \\ &= \lambda(F_1, F_2\#\omega) - \lambda(F_1, F_2) - \lambda(F_1, \omega). \end{aligned}$$

From the paragraph “Cell (2, λ_1)” we know that $\lambda(F_1, F_2\#\omega) - \lambda(F_1, F_2) = 2\text{lk}(\widehat{L}_1\widehat{f}, \mathbf{p}_{1,1})$. Also, by Lemma 4.1, $\lambda(F_1, \omega) = h(w) = h(\mathbf{p}_{1,1})$ and by Claim 2.1, $h(\mathbf{p}_{1,1}) = 0$, so $\lambda(F_1, \omega) = 0$. We get

$$\lambda(F_2, F_1\#\omega) - \lambda(F_2, F_1) = 2\text{lk}(\widehat{L}_1\widehat{f}, \mathbf{p}_{1,1}).$$

Cell $(1, r_1)$. In this cell we need to compute $r_1(\sigma_R(\widehat{f}\#_1\omega_{1,1})) - r_1(\sigma_R(\widehat{f})) = r(F_1\#\omega) - r(F_1)$. Applying Lemma 1.22 we get

$$\begin{aligned} r(F_1\#\omega) - r(F_1) &= r(F_1) + r(\omega) + \frac{\lambda(F_1, \omega) + \lambda(\omega, F_1)}{2} - r(F_1) = \\ &= r(\omega) + \frac{\lambda(F_1, \omega) + \lambda(\omega, F_1)}{2}. \end{aligned}$$

We know that $r(\omega) = 0$ because ω is trivial. Also, $\lambda(F_1, \omega) = 0$, see the end of paragraph “Cell $(1, \lambda_2)$ ”. So

$$r(F_1\#\omega) - r(F_1) = \frac{\lambda(\omega, F_1)}{2} = \frac{l_{1,1}(\widehat{f})}{2}$$

by the definition of $l_{1,1}$.

Cell $(1, r_2)$. Analogous to cell $(2, r_1)$.

Cells $(1, \widehat{W}_1\text{-}\widehat{L}_2)$. Analogous to cells $(2, \widehat{W}_1\text{-}\widehat{L}_2)$.

5. PROOF OF CLAIM 2.8 AND LINKING LEMMA 2.12.

5.1. Proof of Claim 2.8. By Surjectivity Lemma 2.3, there are σ -preimages $[\widehat{f}]$ and $[\widehat{f}']$ of $[f]$ and $[f']$, respectively. The group $H_1(M_1)$ is obtained from $H_1(\widehat{M}_1)$ by adding the relation $[\mathbf{p}_{1,i}] = 0$ for each $1 \leq i \leq m_1$. Since $W_1(f) = W_1(f')$, it follows that $\widehat{W}_1(\widehat{f}') - \widehat{W}_1(\widehat{f}) = \sum_{i=1}^{m_1} a_i [\mathbf{p}_{1,i}]$ for some integers a_i .

Redefine $\widehat{f} := \widehat{f}\#_{i=1}^{m_1} a_i \omega_{1,i}$. By Preimage Lemma 2.9, we still have $\sigma(\widehat{f}) = [f]$. By Calculation Lemma 2.11, we now have $\widehat{W}_1(\widehat{f}) = \widehat{W}_1(\widehat{f}')$. Performing the analogous operation for the remaining three invariants \widehat{L}_1 , \widehat{W}_2 , and \widehat{L}_2 , we can achieve that $\widehat{WL}(\widehat{f}) = \widehat{WL}(\widehat{f}')$. Then $[\widehat{f}]$ and $[\widehat{f}']$ are as required.

5.2. Proof of Linking Lemma 2.12. To prove Linking Lemma 2.12 we shall need the following claim and lemma.

Claim 5.1. *Let $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ be isotopy classes. Then there are embeddings $g_1 : S^3 \rightarrow \text{Int}D_+^6$, $g_2 : S^3 \rightarrow \text{Int}D_+^6$, and $g : S^3 \sqcup S^3 \rightarrow \text{Int}D_+^6$ such that*

- *isotopy classes $[g_1]$ and $[g_2]$ are trivial,*
- *images of g_1 and g_2 are pairwise disjoint and disjoint with the image of \widehat{f} ,*
- *image of g lie in a 6-ball disjoint with the images of \widehat{f} , g_1 , and g_2 ,*
- *$[\widehat{f}\#_1 g_1 \#_2 g_2] \# [g] = [\widehat{f}']$.*

In the special case $\widehat{WL}(f) = \widehat{WL}(f')$ we may choose g so that a simpler equation

$$[\widehat{f}]\#[g] = [\widehat{f}']$$

holds.

Proof. The special case of the claim is proved analogously to part **(II)** of Theorem 1.11. Consider the general case. Analogously to the proof of part **(I)** of Theorem 1.11 we may choose $g_1, g_2 : S^3 \rightarrow \text{Int}D_+^6$ so that $\widehat{WL}(\widehat{f}\#_1 g_1 \#_2 g_2) = \widehat{WL}(f')$. Now apply the special case of the claim to isotopy classes $[\widehat{f}\#_1 g_1 \#_2 g_2]$ and $[f']$. \square

Lemma 5.2. *For any $k \in \{1, 2\}$, $1 \leq i \leq m_k$, and $[\widehat{f}], [\widehat{f}'] \in \widehat{E}^6(\widehat{M}_1 \sqcup \widehat{M}_2)$ the following equality holds*

$$l_{k,i}(\widehat{f}') - l_{k,i}(\widehat{f}) = 2\text{lk}(\mathfrak{p}_{k,i}, \widehat{W}_k(\widehat{f}') - \widehat{W}_k(\widehat{f})).$$

Proof. Let $g_1 : S^3 \rightarrow \text{Int}D_+^6$, $g_2 : S^3 \rightarrow \text{Int}D_+^6$, and $g : S^3 \sqcup S^3 \rightarrow \text{Int}D_+^6$ be embeddings as in the statement of Claim 5.1. Denote

- by F , F' , and G' the restrictions of $\sigma_R(\widehat{f})$, $\sigma_R(\widehat{f}')$, and g to the k -th component, respectively,
- $G := g_k$,
- $\omega := \omega_{k,i}$.

By Claim 5.1 we have $[F'] = [F\#G\#G']$. The isotopy between F' and $F\#G\#G'$ is fixed on $\omega(S^3) \subset D_-^6$ so without a loss of generality we may assume that $F' = F\#G\#G'$.

By the definition of $l_{k,i}$ we have

$$(1) \quad l_{k,i}(\widehat{f}') - l_{k,i}(\widehat{f}) = \lambda(\omega, F\#G\#G') - \lambda(\omega, F) = \lambda(\omega, F\#G) - \lambda(\omega, F)$$

where the last equality holds because the image of G' lies in a 6-ball in D_+^6 disjoint from the images of ω , F , and G .

Let us compute $\lambda(\omega, F\#G)$. Next two equalities follow from Lemma 1.22

$$\begin{aligned} 2r(\omega\#F\#G) &= \lambda(\omega, F\#G) + \lambda(F\#G, \omega) + 2r(\omega) + 2r(F\#G), \\ 2r(\omega\#F\#G) &= \lambda(F, \omega\#G) + \lambda(\omega\#G, F) + 2r(\omega\#G) + 2r(F). \end{aligned}$$

We get

$$\lambda(\omega, F\#G) = \lambda(F, \omega\#G) + \lambda(\omega\#G, F) + 2r(\omega\#G) + 2r(F) - \lambda(F\#G, \omega) - 2r(\omega) - 2r(F\#G).$$

Clearly, ω is trivial so $r(\omega) = 0$. Also, G is trivial by Claim 5.1. Moreover, image of ω is in the boundary of D_+^6 while the image of G is in the interior of D_+^6 . So, $r(\omega\#G) = 0$. Now we can simplify the formula for $\lambda(\omega, F\#G)$ above to get

$$\lambda(\omega, F\#G) = \lambda(F, \omega\#G) + \lambda(\omega\#G, F) + 2r(F) - \lambda(F\#G, \omega) - 2r(F\#G).$$

By Lemma 1.22, we have $2r(F\#G) = 2r(F) + 2r(G) + \lambda(F, G) + \lambda(G, F) = 2r(F) + \lambda(F, G) + \lambda(G, F)$. So

$$\lambda(\omega, F\#G) = \lambda(F, \omega\#G) + \lambda(\omega\#G, F) - \lambda(F\#G, \omega) - \lambda(F, G) - \lambda(G, F).$$

By Lemma 1.20, we have $\lambda(\omega\#G, F) = \lambda(\omega, F) + \lambda(G, F)$ and $\lambda(F\#G, \omega) = \lambda(F, \omega) + \lambda(G, \omega) = \lambda(F, \omega)$, where the last equality holds because the images of ω and G lie in disjoint 6-balls meaning that $\lambda(G, \omega) = 0$. So

$$\lambda(\omega, F\#G) = \lambda(F, \omega\#G) + \lambda(\omega, F) - \lambda(F, \omega) - \lambda(F, G).$$

Going back to equation (1) we get

$$l_{k,i}(\widehat{f}') - l_{k,i}(\widehat{f}) = \lambda(F, \omega\#G) - \lambda(F, \omega) - \lambda(F, G).$$

Let $\Delta_G \subset \text{Int}D_+^6$ be an embedded framed disk bounded by $G(S^3)$. Denote

$$d := F^{-1}(F(S^3) \cap \Delta_G)$$

and

$$w := F^{-1}(F(S^3) \cap \Delta_{\omega, k, i}).$$

By Lemma 4.1 we have

- $\lambda(F, \omega\#G) = h(w \sqcup d)$,
- $\lambda(F, \omega) = h(w)$,

- $\lambda(F, G) = h(d)$.

So

$$\begin{aligned} l_{k,i}(\widehat{f}') - l_{k,i}(\widehat{f}) &= h(w \sqcup d) - h(w) - h(d) = \\ &= h(w) + h(d) + 2\text{lk}(w, d) - h(w) - h(d) = 2\text{lk}(w, d) = 2\text{lk}(\mathfrak{p}_{k,i}, \widehat{W}_k(\widehat{f}') - \widehat{W}_k(\widehat{f})). \end{aligned}$$

The last equation holds because

- $w = \mathfrak{p}_{k,i}$ by Claim 4.2,
- $d \subset \text{Int}M_k$ is a representative of the homology class $\widehat{W}_k(\widehat{f}') - \widehat{W}_k(\widehat{f}) \in H_1(\widehat{M}_k)$ by the definition of \widehat{W}_k .

□

Proof of Linking Lemma 2.12. By Lemma 5.2, it is enough to prove the lemma in the special case $\widehat{f} = \widehat{f}^0$. I.e., we need to prove that

$$\sum_{i=1}^{m_k} a_i[\mathfrak{p}_{k,i}] = 0 \quad \Rightarrow \quad \sum_{i=1}^{m_k} a_i l_{k,i}(\widehat{f}^0) = \sum_{i=1}^{m_k} 2\text{lk}(\mathfrak{p}_{k,i}, \widehat{W}_k(\widehat{f}^0)).$$

The righthand side is zero because $\widehat{W}_k(\widehat{f}^0) = 0$ by definition. Therefore we need to prove that

$$\sum_{i=1}^{m_k} a_i[\mathfrak{p}_{k,i}] = 0 \quad \Rightarrow \quad \sum_{i=1}^{m_k} a_i l_{k,i}(\widehat{f}^0) = 0.$$

Consider the embedding $\widehat{f}' := \widehat{f}^0 \#_{i=1}^{m_k} a_i \omega_{k,i}$.

We have that

$$\widehat{W}_k(\widehat{f}') = \widehat{W}_k(\widehat{f}') - \widehat{W}_k(\widehat{f}^0) = \sum_{i=1}^{m_k} a_i[\mathfrak{p}_{k,i}] = 0,$$

where the first equation holds because $\widehat{W}_k(\widehat{f}^0) = 0$ and the second equation holds by Calculation Lemma 2.11. Also by Calculation Lemma 2.11, we get that the rest of the Whitney invariants of \widehat{f}' and \widehat{f}^0 are also the same, namely $\widehat{WL}(\widehat{f}') = \widehat{WL}(\widehat{f}^0) = 0$.

By Claim 5.1 (the “special case”), there is an embedding $g : S^3 \sqcup S^3 \rightarrow S^6$ such that

$$(2) \quad [\widehat{f}^0] \# [g] = [\widehat{f}'].$$

On one hand, from the commutativity of the action $\#$ (Claim 2.2) we get

$$r_k(\sigma_R \widehat{f}') - r_k(\sigma_R \widehat{f}^0) = r_k(g).$$

On the other hand, by Calculation Lemma 2.11, we get

$$r_k(\sigma_R \widehat{f}') - r_k(\sigma_R \widehat{f}^0) = \sum_{i=1}^{m_k} a_i \frac{l_{k,i}(\widehat{f}^0)}{2}.$$

So

$$\sum_{i=1}^{m_k} a_i \frac{l_{k,i}(\widehat{f}^0)}{2} = r_k(g).$$

It remains to prove that $r_k(g) = 0$.

By the commutativity of the action $\#$, we get from (2) that

$$\sigma([\widehat{f}']) = \sigma([\widehat{f}^0]) \# [g].$$

On the other hand, by Preimage Lemma 2.9, we have

$$\sigma([\widehat{f'}]) = \sigma([\widehat{f^0}]).$$

Therefore,

$$\sigma([\widehat{f^0}])\#[g] = \sigma([\widehat{f^0}]).$$

Consider the restriction of $\sigma([\widehat{f^0}])$ to M_k . Its Whitney invariant W is equal to $W_k(\sigma\widehat{f^0}) = W_k(f^0) = 0$. So $r_k(g) = 0$ by Theorem 1.3, part **(III)**. \square

REFERENCES

- [Av16] S. Avvakumov. *The classification of certain linked 3-manifolds in 6-space*. Moscow Mathematical Journal 16.1 (2016): 1–25.
- [BH70] J. Boéchat, A. Haefliger. *Plongements différentiables des variétés orientées de dimension 4 dans \mathbb{R}^7* . Essays on topology and related topics. Springer Berlin Heidelberg, 1970. 156–166.
- [Bo71] J. Boéchat. *Plongements de variétés différentiables orientées de dimension $4k$ dans \mathbb{R}^{6k+1}* . Commentarii Mathematici Helvetici 46.1 (1971): 141–161.
- [CS16a] D. Crowley, A. Skopenkov. *Embeddings of non-simply-connected 4-manifolds in 7-space. I. Classification modulo knots*. arXiv preprint arXiv:1611.04738 (2016).
- [CS16b] D. Crowley, A. Skopenkov. *Embeddings of non-simply-connected 4-manifolds in 7-space. II. On the smooth classification*. arXiv preprint arXiv:1612.04776 (2016).
- [Ha62a] A. Haefliger. *Differentiable links*. Topology 1.3 (1962): 241–244.
- [Ha62b] A. Haefliger. *Knotted $(4k-1)$ -spheres in $6k$ -space*. Annals of Mathematics (1962): 452–466.
- [Ha66] A. Haefliger. *Enlacements de sphères en codimension supérieure 2*. Commentarii Mathematici Helvetici 41.1 (1966): 51–72.
- [Ha67] A. Haefliger. *Lissage des immersions–I*. Topology 6.2 (1967): 221–239.
- [Hi61] M. Hirsch. *The imbedding of bounding manifolds in Euclidean space*. Annals of Mathematics (1961): 494–497.
- [Hu70] J. F. P. Hudson. *Concordance, isotopy, and diffeotopy*. Annals of Mathematics (1970): 425–448.
- [Hu72] J. F. P. Hudson. *Embeddings of bounded manifolds*. Mathematical Proceedings of the Cambridge Philosophical Society. Vol. 72. No. 01. Cambridge University Press, 1972.
- [MAa] *Manifold atlas. Embeddings in Euclidean space: an introduction to their classification*. http://www.map.mpin-bonn.mpg.de/Embeddings_in_Euclidean_space:_an_introduction_to_their_classification
- [MAb] *Manifold atlas. Knots, i.e. embeddings of spheres*. http://www.map.mpin-bonn.mpg.de/Knots,_i.e._embeddings_of_spheres
- [Mi65] J. Milnor. *Lectures on the h-cobordism theorem*. Princeton University Press, 1965.
- [PS97] V. Prasolov, A. Sossinsky. *Knots, links, braids, and 3-manifolds: an introduction to the new invariants in low-dimensional topology*. No. 154. American Mathematical Soc., 1997.
- [Sk08a] A. Skopenkov. *A classification of smooth embeddings of 3-manifolds in 6-space*. Mathematische Zeitschrift 260.3 (2008): 647–672.
- [Sk08b] A. Skopenkov. *Embedding and knotting of manifolds in Euclidean spaces*. London Mathematical Society Lecture Note Series 347 (2008): 248.
- [Sk15] A. Skopenkov. *Classification of knotted tori*. arXiv preprint arXiv:1502.04470v1 (2015).
- [Vr77] J. Vrabec. *Knotted a k -connected closed PL m -manifold in \mathbb{R}^{2m-k}* . Transactions of the American Mathematical Society (1977): 137–165.
- [Wh61] J. H. C. Whitehead. *Manifolds with transverse fields in euclidean space*. Annals of Mathematics (1961): 154–212.
- [Ze63] E. C. Zeeman. *Unknotting combinatorial balls*. Annals of Mathematics (1963): 501–526.
- [Ze93] E. C. Zeeman. *A brief history of topology*. UC Berkeley, October 27, 1993, On the occasion of Moe Hirsch’s 60th birthday.
- [Zh16] A. V. Zhubr. *On smoothing embeddings and isotopies*. Mathematical Notes 99.5–6 (2016): 946–947.